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CONFIRMATORY EXPERIMENTS ON THE VALUE OF THE SOLAR CONSTANT OF RADIATION

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We have made hitherto nearly 1000 determinations of the intensity of solar radiation outside the atmosphere at mean solar distance, termed the solar constant of radiation. The mean value found is 1.93 calories per square centimeter per minute. Langley's spectro-bolometric method was employed. This consists in determining the distribution of the energy in the solar spectrum at different solar zenith distances, and thereby computing coefficients of atmospheric transmission suitable to determine the energy curve outside the atmosphere. The bolometric measurements are reduced in terms of standard 15° calories per square centimeter per minute by the aid of comparisons made each day of observation with standardized pyrheliometers. Observations have been made at Washington (sea level); Bassour, Algeria (1160 meters); Mount Wilson, California (1730 meters); and Mount Whitney, California (4420) meters. They have continued during all the years 1903 to 1914. Great changes from day to day and from place to place in temperature, in barometric pressure, in humidity, in haziness, while of course greatly affecting measurements of intensity at the stations, and of atmospheric transparency computed, nevertheless have not produced differences of the solar constant values. This seems to us to be strong evidence of the soundness of the method.

In the second place it has been shown by Fowle that the atmospheric transmission coefficients obtained at Mount Wilson fit well with Lord Rayleigh's theory of atmospheric scattering, except for those regions of spectrum where numerous atmospheric lines and bands of true absorp-

tion are known to occur. Fowle has computed from the transmission coefficients that the number of molecules per cubic centimeter of air at standard temperature and pressure is $(2.70 \pm 0.02) \times 10^{19}$. This value is very close to Millikan's determination by absolutely independent observations and methods, namely $(2.705 \pm 0.005) \times 10^{19}$.

In the third place simultaneous solar-constant observations at Mount Wilson and Bassour, separated by one-third the earth's circumference, unite in showing a substantial irregular variability of the sun from day to day. This solar variability has been of late independently confirmed by us by examination of the distribution of brightness along the diameter of the sun's disk. The latter observations show variations of distribution from day to day, and these accompany pretty closely the variations of the total solar radiation. It seems to us that, as the fact of solar variability is thus independently confirmed as a real phenomenon, it speaks favorably for the substantial accuracy of our solar constant measurements that it was through them that the irregular variations of from 1 to 5 or, very rarely, 10% were first discovered.

Notwithstanding these evidences of the soundness of our solar constant work, various attacks upon it have been made, tending to show that the solar constant is much higher than 1.93 calories, perhaps even 3.5 to 4.0 calories. A principal argument is that the atmospheric transparency continually diminishes as the sun rises within 75° zenith distances, so that our values of atmospheric transmission are much too great, and have no relation to the transmission of an atmosphere of constant transparency. Secondly it is said that measurements of solar radiation exceeding 1.93 calories have been made on mountain tops, and from free balloons. Various other objections are raised, which we discuss in our paper now being published by the Smithsonian Institution.

On two days, September 20 and 21, 1914, we continued solar constant observations at Mount Wilson from the instant of sunrise until about 10 o'clock. We have reduced the work by the aid of Bemporad's air-mass formulae and tables. As these postulated uniform optical quality of the atmosphere from bottom to top, it was necessary to apply certain corrections to them varying with the wave-length, depending upon the extinction by water vapor residing in the lowest atmospheric strata. We were enabled to determine these corrections by Fowle's studies of the effects of water vapor. We find on both days that the atmospheric transparency remained sensibly unaltered from sunrise to 10 o'clock. Closely identical values of the solar constant are obtained, whatever the range of air masses used to determine the atmospheric transmission. We made three independent estimates for each

day, for air-mass ranges 1.3 to 4; 4 to 12; and 1.3 to 20 air-masses, respectively. All six solar constant values thus found fall between 1.90 and 1.95 calories. The smallest air masses, as it happens, yield slightly the highest values. We conclude that our previous results have not been made too small by neglecting to observe during the time when the sun is within 15° of the horizon.

On July 11, 1914, in coöperation with the United States Weather Bureau, a recording pyrheliometer attached to sounding balloons was sent up to the altitude of about 24 km., where the barometric pressure was 3 cm. of mercury, which is only one twenty-fifth of that which prevails at sea level.

Good records of solar radiation were obtained over a period of more than an hour, and including the period when the instrument reached highest elevation. The mean value of the best three records made at highest altitudes, as reduced to mean solar distance, comes out 1.84 calories per cm.^{2} per minute. We believe about 2% should be added to represent radiation scattered and absorbed in the atmosphere above the level reached, making for the probable value of the solar constant, from this day's work, 1.88 calories. This value falls decidedly within the range of solar constant values we have observed. We state in connection with it the following results which are the highest reliable direct observations of solar radiation at the various altitudes, as reduced to mean solar distance and vertical sun:

Station	*Washington*	*Mount Wilson*	*Mount Whitney*	*Manned Balloon*	*Free Balloon*
Altitude.....	127 m.	1730 m.	4420 m.	7500 m.	24000 m.
Barometer.....	75 cm.	62 cm.	45 cm.	30 cm.	3 cm.
Radiation	1.58	1.64	1.72	1.755	1.84 cal.
Observer	Kimball	Abbot	Abbot	A. Peppler	(Smithsonian)

VARIATION OF FLOWER SIZE IN NICOTIANA

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During the past five years considerable attention has been given to the study of the inheritance of flower-size in *Nicotiana* at the University of California Botanical Gardens. In the course of these investigations it has been found that flower size varies markedly under different conditions attending development. This report is concerned with an analysis of some of these conditions, and the bearing of such variations on the study of flower-size inheritance in *Nicotiana*.

The conclusions drawn in this paper are based on some 25,000 measurements taken on the following pure lines and hybrids of *Nicotiana:*

1. *N. Tabacum* var. *macrophylla*[1] in the growing season 1913. This *N. Tabacum* variety has been grown at the University of California Botanical Gardens for six years in the pure line, under the garden number 22/07.

2. The F_1 hybrid, *N. Tabacum* var. *macrophylla* × *N. sylvestris*,[2] which will be referred to as F_1H_{38}. Flowers of this hybrid were measured on plants from the first sowing of the hybrid seed in 1912, on the same plants cut back and coming up from the roots in 1913, and on a second sowing of the same hybrid seed in 1913.

3. F_1H_{137}, the reciprocal of the above cross, flowers of which were measured in 1913.

4. The F_1 hybrid produced by crossing the F_1 of the hybrid *N. Tabacum* var. Maryland × *N. Tabacum* var. Cavala with *N. sylvestris*, the garden number of which is H_{44}. As with H_{38}, flowers of this hybrid were first measured in 1912, and again on the same plants cut back and coming up from the roots in 1913. In addition cuttings of these plants were made and measurements taken on them.

From the data on these and other populations it has been found possible to determine a number of factors which influence flower size in *Nicotiana*. For convenience these are treated under various rather arbitrarily selected headings. Under the 'age of plant' heading are included not only a consideration of the difference in the size of flowers borne early in the season as compared with those borne late in the season on the same plants, but also a consideration of the difference in the size of flowers the first blooming season of the plant compared with the size of the flowers produced the second year on the same plants cut back and coming up from the roots. Under the 'age of flower' heading are included first, a consideration of the difference in the size of flowers borne on the terminal inflorescences just going out of flower and those borne at the same time on laterals and second, the influence of age on the individual flower; i.e., comparisons of measurements of flower fully opened, before and after shedding pollen. Other factors such as the influence of the removal of flowers and developing seed capsules, the behavior of cuttings under various conditions, and the influence of soil fertility have likewise furnished data upon this variability of flower size under conditions attending development.

When plants first come into flower the spread and length of corolla are greater than the spread and length of flowers produced on the same plants later in the growing season. Thus the mean spread of corolla

of 75 plants of *N. Tabacum* var. *macrophylla* at the beginning of the blooming period was 28.49 ± 0.09 mm. and the length 45.60 ± 0.07 mm. Measurements of flowers taken on an average one month later gave a mean corolla spread for the same plants of 24.19 ± 0.22 mm. and for length 42.25 ± 0.15 mm. For the same plants means derived by grouping together the measurements of 25 of these plants early in the season, 25 in the middle of the period, and the remaining 25 toward the end of the period of measurement gave for spread 26.69 ± 0.20 mm. and for length 43.96 ± 0.18 mm., which figures lie very close to the means for all the plants for the entire season which are 26.83 ± 0.09 mm. and 44.27 ± 0.07 mm. respectively.

The latter arrangement brings up a practical matter of considerable importance in connection with genetic investigations of flower size in *Nicotiana*. In this case although the means were practically identical for the two arrangements, the coefficient of variability where each third of the plants was measured at a different time was 9.77 ± 0.54% whereas that based on the distribution of means for all the plants for the entire season was less than half as great, 4.45 ± 0.24%. For length the corresponding coefficients of variability, 5.18 ± 0.29% and 2.13 ± 0.12% respectively, show a closely proportionate correspondence in this respect. This higher variability is exactly what would be obtained if a large F_2 population were measured progressively, that is beginning at one end of the field and working through until the last plant had been measured. Such a method of measurement used in comparing large F_2 populations with relatively small parent and F_1 populations might very easily lead to erroneous conclusions on the critical point of the experiment, namely the extent of increase of variability in F_2.

In the case of the semi-sterile hybrids of *N. Tabacum* varieties with *N. sylvestris* another factor enters in, namely the lack of seed production and the consequent prolonged blooming period. There is, therefore, not so sudden a decrease in corolla size as in the case of *N. Tabacum* var. *macrophylla;* but nevertheless a marked decrease occurs rather later in the blooming season. In the case of spread of flowers of F_1H_{137} there was a decrease during the period from August 9 to September 19 from 38.33 ± 0.11 mm. to 31.67 ± 0.26 mm. For length the decrease was less striking, from 56.22 ± 0.09 mm. to 53.44 ± 0.28 mm. Similarly in 13 (11) F_1H_{38} there was a decrease in corolla spread from August 4 to November 7 from 39.10 ± 0.31 mm. to 33.10 ± 0.22 mm., and in length from 57.10 ± 0.22 mm. to 52.60 ± 0.29 mm. Flowers of $13F_1H_{38}$, genetically the same but a different population blooming the second

year on their own roots, were measured at the beginning of the flowering season during the period from May 21 to June 20. During this time the flower size did not decrease as is shown by comparing the mean spread on the first dates of 38.81 ± 0.31 mm. with 40.05 ± 0.35 mm., the mean spread for the last dates, and in the case of length of corolla, 55.19 ± 0.15 mm. and 53.81 ± 0.24 mm., a slight decrease. On Plant 6, however, flowers were measured again on July 31. It was found that a mean spread of 43.41 mm. based on measurements of all the flowers on June 16 had decreased to 37.38 mm. based on measurements of all the flowers on July 31. H_{44}, which was measured during this same period, similarly shows only a slight decrease in flower size, but in every case where flowers were measured on the later date a striking decrease in size was obtained. We have taken the results of similar determinations and condensed them in the following table in which not only the means and their probable errors are given, but also the differences between measurements of different dates and the probable errors of these differences. These differences are to be regarded as significant in case they exceed two and a half or three times the corresponding probable error. In each case the constant for spread is given on the first line, that for length on the second.

Influence of Age of Plant on Flower Size. Measures in Millimeters

GARDEN NUMBER	MEANS FOR DATES			DIFFERENCES BETWEEN DATES		
	First	Mid	Last	First and Mid	Mid and Last	First and Last
$13F_1H_{44}$	38.53 ± 0.38	40.16 ± 0.44	39.32 ± 0.64	−1.63 ± 0.58	0.84 ± 0.77	−0.79 ± 0.74
	54.05 ± 0.26	54.53 ± 0.28	52.84 ± 0.49	−0.48 ± 0.39	1.69 ± 0.57	1.21 ± 0.56
$13F_1H_{33}$	38.81 ± 0.31	39.05 ± 0.40	40.05 ± 0.35	−0.24 ± 0.51	−1.00 ± 0.53	−1.24 ± 0.47
	55.19 ± 0.15	55.43 ± 0.15	53.81 ± 0.24	−0.24 ± 0.21	1.62 ± 0.28	1.38 ± 0.28
$13(11)F_1H_{33}$	39.10 ± 0.31	36.10 ± 0.43	33.10 ± 0.22	3.00 ± 0.53	3.00 ± 0.49	6.00 ± 0.38
	57.10 ± 0.22	54.10 ± 0.29	52.60 ± 0.29	3.00 ± 0.37	1.50 ± 0.41	4.50 ± 0.37
$13F_1H_{137}$	38.83 ± 0.11	36.67 ± 0.44	31.67 ± 0.26	1.66 ± 0.45	5.00 ± 0.51	6.66 ± 0.28
	56.22 ± 0.09	55.00 ± 0.18	53.44 ± 0.28	1.22 ± 0.21	1.56 ± 0.34	2.78 ± 0.30
13 22/07	28.48 ± 0.09		24.19 ± 0.22			4.29 ± 0.24
	45.60 ± 0.07		42.25 ± 0.15			3.35 ± 0.17

With respect to the influence of removal of flowers on corolla size, the measurements show a striking correspondence between the size of corolla and the presence of developing seed capsules on the plant. By removing all flowers from the plant as fast as they go by it is possible to keep up the flower size to nearly that of the first flowers produced, and in some cases to double the life of the plant. A case in point is that of two plants of the *N. Tabacum* var. *macrophylla* series. On Plant 14 only the flowers measured were removed, while on Plant 15 all flowers

too old to be measured and all developing seed capsules were removed twice a week. In a month the spread of corolla for Plant 14 decreased 5.78 mm. and for Plant 15 only 0.75 mm. At the end of two months the decrease amounted to 5.40 mm. for Plant 14 and 2.29 mm. for Plant 15. The length behaved similarly in this case, in Plant 14 there was a decrease of 4.55 mm. and in Plant 15 of 0.25 mm.; and at the end of two months, a decrease of 4.00 mm. in Plant 14 and 0.79 mm. in Plant 15.

During the period which elapses from the time a flower is fully opened to the time at which pollen is shed, there is a considerable increase in corolla spread and associated with it little or no increase in corolla length. On ten plants of F_1H_{38}, the increase in spread amounted on an average to 4.53 ± 0.23 mm. and for length there was an increase of 1.62 ± 0.22 mm. Similarly the spread of corolla for F_1H_{137} averaged 3.18 ± 0.17 mm. greater for flowers after shedding pollen, but in this case the length was 0.47 ± 0.11 mm. smaller, a slight discrepancy undoubtedly due to the indirect method of comparison made, which still further emphasizes the increase in spread of corolla during this period. This increase in spread has, also, been repeatedly confirmed on individual marked flowers.

That there is a differential distribution of flowers on tobacco plants according to size at any given time is shown by the comparative measurements of flowers borne among developing seed capsules on the terminal inflorescence of a plant and those borne on laterals of the same plants. In the case of plants of *N. Tabacum* var. *macrophylla* the flowers borne on laterals were found to have an average spread greater by 2.56 ± 0.16 mm. than those of the terminal inflorescence, and in the case of length, 1.24 ± 0.11 mm. greater. In the hybrids studied the difference is rather more marked, so much so that there is a distinctly noticeable difference in size between these two classes of flowers.

Similarly we have found that numerous other factors have a like influence on flower size, some relatively great and others less marked. For instance cuttings of F_1H_{44} growing in the greenhouse produced flowers 3.95 mm. smaller in spread and 1.42 mm. greater in length than those on the field plants from which the cuttings were taken. Pure line populations of *N. sylvestris* grown in a shaded situation on rich moist garden soil produced distinctly larger flowers than plants growing on higher unfertilized soil and not shaded, and pot experiments likewise showed that flower size could be distinctly influenced by applications of sodium nitrate, and in a direction parallel to that of the influence on vegetative characters.

The conclusion seems irresistible that flower-size in *Nicotiana* is not

so constant as it has been assumed to be, but that it is affected by a number of conditions, and that at least some of these may not affect length and spread in the same manner. Attention has been called to these facts because they have not been given adequate consideration in genetic research on the behavior of flower size in *Nicotiana* and other genera.

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[1] Setchell, Studies in *Nicotiana*, I, Univ. of Cal. Pub. Bot., 5, 8, 1912.
[2] Ibid., p. 29.

RETENTION IN THE CIRCULATION OF DEXTROSE IN NORMAL AND DEPANCREATIZED ANIMALS, AND THE EFFECT OF AN INTRAVENOUS INJECTION OF AN EMULSION OF PANCREAS UPON THIS RETENTION

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The content of dextrose in the circulating blood of normal animals is almost constant, it amounts to about 0.1 percent. The carbohydrates of the foodstuffs form the main source of dextrose in the body. On their way from the digestive tract the carbohydrates are transformed into various forms of saccharides; but all are finally converted largely into glycogen, which is stored up mostly in the liver. The blood obtains its supply of dextrose from the glycogen of the liver, and distributes it among the tissues of the body according to their demand for it. In the normal animal none of the dextrose escapes through the kidneys. Accordingly the constancy of the amount of dextrose in the blood is regulated by a mechanism which controls either of the two factors; the ***supply*** of or the ***demand*** for it.

In diabetes the dextrose content of the blood is higher than normal, is variable in amount, and, when it is sufficiently high, dextrose escapes through the kidneys. The cause of the increase of the blood dextrose, or hyperglycaemia, may be found either in a decrease in the demand of the tissues for dextrose, that is, the tissues burn dextrose less readily than in normal conditions; or in an increase in the supply, that is, the liver supplies the blood with more dextrose than in normal conditions.

Both possibilities have groups of followers who have brought forward numerous experiments in support of either of the respective theories.

It was experimentally established years ago that the complete removal of the pancreas brings on a severe hyperglycaemia, leading to considerable glycosuria and to the death of the animal. Here we are confronted again with the question, whether the presence of the pancreas in the normal animal favors the burning of the dextrose by the tissues, or whether it restricts the supply of dextrose to the blood by the liver. But there also arose another question, namely, whether the effect of the removal of that gland was due to a removal of a controlling substance, or, whether it was due, in consequence of the operation, to an interference with the nervous mechanism which controls the supply of the dextrose. In case of the first alternative, namely, that the resulting hyperglycaemia is due to a removal of some substance, normally secreted by the pancreas, we have to assume that that substance belongs to that group of substances which are provided by glandular and other tissues for the proper maintenance of functions of the animal body, i.e., *internal secretions*, since a complete suppression of the external secretions of the pancreas never leads to hyperglycaemia or diabetes.

We wish to report briefly results of several series of experiments which are likely to shed some light upon these questions. Most of our experimental series have one feature in common, namely, that the blood was provided with dextrose directly and not by way of the liver. In one series of experiments large quantities of dextrose (20 cc. of a 20% solution of dextrose per kilo body weight) were injected intravenously into normal dogs. It was found by us, as it was found by some investigators before us, that the injected large surplus of dextrose disappeared quite rapidly from the circulation. *Ninety minutes after the infusion, the dextrose content of the blood reached nearly the same level which it had before the injection.* About one-half of the injected quantity of dextrose left the circulation through the kidneys; the other half went into the tissues. In nephrectomized animals also the dextrose returned to the previous level in the same length of time. *Here we have a striking example of the ability and readiness of the circulation of a normal animal to rid itself of a large surplus of dextrose.*

In another series of experiments similar injections of dextrose were made into animals, the pancreas of which was removed by an operation a day or two before the dextrose infusion. The sample of blood taken from the animal before the intravenous injection of dextrose was strongly hyperglycaemic. It was then found that in these experiments *the dextrose content of the sample of blood, which was taken ninety minutes after*

the end of the dextrose injection, was at least twice as high as that of the sample taken before the injection. Apparently the removal of the pancreas interfered with the power of the circulation to rid itself readily of a surplus of dextrose. The possible ability of the liver to provide the circulation with a greater supply of dextrose in the absence of the pancreas can certainly be no important factor in these results. Our experiments show, then, *that in the presence of the pancreas the circulation rids itself easily of intravenously introduced dextrose; but that it is unable to do it satisfactorily in the absence of the pancreas.* On the basis of these facts is it not plausible to assume further that the pancreas exerts exactly the same influence, when the dextrose in the circulation is being supplied in some other manner than by a burette into a vein, for instance, by the liver through its connections with the circulation? Or, in other words, do not our experiments make the assumption plausible that pancreatic hyperglycaemia is due to the fact that *in the absence of the pancreas the circulation is unable to dispose properly of the dextrose which it receives in normal amounts from the liver?*

In a third series, the experiments were again made by injecting dextrose intravenously in depancreatized dogs. But in these cases a strained pancreatic emulsion was added to the dextrose solution, and in some instances the infusion of the pancreatic emulsion was continued for some time after the infusion of the dextrose was finished. We present here merely a preliminary communication; we have not yet mastered all the details of this part of the investigation. But there seems to be no doubt about the nature and validity of the main result, which is this: In such experiments, *ninety minutes after the end of the dextrose infusion, the dextrose content of the blood is again at about the same level as it was before the infusion of the dextrose.* Apparently the presence of a pancreatic emulsion within the blood helps the circulation to get rid of the surplus of dextrose injected intravenously. In these experiments it is evident that the *effect of the pancreas can be only of a chemical and not of a nervous nature,* or, in other words, it is the internal secretion of the pancreas which helps the circulation to get rid of the surplus of dextrose.

To the above results we wish now to add the following preliminary statement derived from three experiments. *In depancreatized animals the blood of which has shown a marked hyperglycaemia, an intravenous infusion of a pancreas emulsion brought the dextrose content of the blood to 0.09% and even 0.08%.* The hyperglycaemia returned next day and could be reduced again to the normal level by an infusion of pancreas emulsion.

Conclusions.—In normal animals the circulation possesses the ability to get rid readily of a surplus of dextrose injected intravenously. In the absence of the pancreas this ability of the circulation is impaired. This ability can be temporarily restored by an intravenous injection of a pancreas emulsion.

Furthermore, an intravenous injection of a pancreas emulsion is capable of reducing the hyperglycaemia due only to depancreatization to a normal level of the dextrose content of the blood.

As to the nature of the factors which may constitute the ability or inability of the circulation to get rid of a certain degree of surplus of the dextrose content of the blood, we are not willing to discuss it at this stage of our investigation. We are rather bent upon seeing how many more facts we shall be enabled to bring to light on the basis of the hypothesis which set us to work on these problems.

PARTHENOCARPY AND PARTHENOGENESIS IN NICOTIANA

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The occurrence of parthenogenesis in the genus *Nicotiana* has, in general, been considered to be a negligible factor so far as the interpretation of breeding experiments with tobacco is concerned. The only outstanding instance in which castrated or mutilated tobacco flowers have yielded viable seed is to be found in a report describing the experiments of Mrs. R. H. Thomas.[1] Her experiments, apparently conducted with due regard to the various sources of error, indicate that for her cultures parthenogenesis in various species and hybrids of *Nicotiana* is of frequent occurrence. Conflicting evidence is furnished by the experiments of others. Thus, Howard,[2] following experiments which involved the emasculation of over 5000 flowers on many strains of Indian Tobacco, found but five capsules containing seed and in only two cases was it shown that this seed was viable. Hartley[3] obtained two capsules of seed 'by treating fully receptive stigmas with magnesium sulphate,' employing in his experiments flowers of 'Cuban Tobacco (*Nicotiana Tabacum*).' The seeds thus produced proved, however, to be nothing more than empty shells. Further, East[4] and Wellington[5] claim that hybrid seed, produced by crosses between certain species of *Nicotiana*, has given plants 'like the mother species and also true hybrids,' plants 'like the mother species and no true hybrids,' and that this seed gave 'no true hybrids on one occasion but did produce true hybrids on other occa-

sions.' Certain species-crosses in *Nicotiana* made by Gärtner, also, gave seed that produced the mother species as well as true hybrids. These latter observations have been explained by the assumption that cross pollinations, 'by the extraordinary irritation of foreign pollen' (East, l.c.), may induce an apogamic or parthenogenetic development of a portion of or all of the ovules affected. Wellington, by a variety of stimulations and irritations of immature tobacco flowers was able to produce 'some abortive seed without embryos' while from a considerable number of simple castrations of buds, viable seed was produced in only one doubtful case. Our own results, based upon over 1500 castrations of tobacco flowers, borne on many species and varieties of *Nicotiana* in the University of California Botanical Garden, are entirely negative. No viable seed without pollination has been produced and in the appearance of numerous hybrid progenies we have, in no case, had reason to assume the production of apogamic or parthenogenetic seed as a result of cross pollination.

On the assumption, then, that errors of technique will not entirely account for the results which Mrs. Thomas obtained, it appeared possible that the production of viable seed without pollination was actually peculiar to certain of the particular strains of tobacco which she possessed. Faulty technique has been the only explanation offered to account for the unusual results which Mrs. Thomas reported while, so far as I know, no one has attempted to repeat her experiments with the strains of tobacco she used. Seed of the *Nicotiana Tabacum* variety, described in her paper as a frequent producer of seed without pollination, was kindly sent me by Mrs. Thomas. On plants from this same seed Bateson[6] confirmed Mrs. Thomas' results. The seed was received under the name '*Nic. tabaccum* Cuba' and represented a portion of the original seed 'gathered in the Garden of Casa Loring at Malaga in 1908.' Ninety-five plants were grown from this seed during 1914. The stand was entirely uniform and the plants were taller than and as vigorous as any of the *N. Tabacum* varieties in our cultures (cf. Setchell[7]). As described by Mrs. Thomas (loc. cit.) they are white-flowered plants, considerable numbers of four-parted flowers being produced especially at the opening of the flowering season. In vegetative characters they resemble the commercial types collectively known as 'Cuban Tobacco.'

Eight hundred buds on these ninety-five plants of '*Nic. tabaccum* Cuba' were treated according to one of the following methods—(1) simple emasculation of the flower by picking off the anthers near the tops of the filaments, (2) castration of the bud plus the pinching off, with the forceps, of the stigma at the very top of the style and (3) pinch-

ing off the stigma in the bud without the removal of the anthers. The crushed tip of the style in such cases dries down considerably before the flower matures or pollen is shed. The last two types of treatment, (2) and (3), will be referred to as 'mutilations.' In addition a few decapitation experiments were attempted (cf. Thomas, l.c.). The number of simple emasculations far exceeded the other two types of treatment.

The length of every bud treated was noted by a measurement from the point of union of calyx and pedicel to the tips of the folded corolla lobes. The maximum length of unopened buds was 49 mm. and, with the exception of a few late flowers, the anthers do not open until the corolla lobes begin to open back. The number of the flower parts was, also, determined for each bud on the supposition that four-parted and five-parted flowers might give different results following emasculation or mutilation. Finally, the records were so kept that the position on the plant of a given treated bud can be determined from the field notes—i.e., whether it occurred on the terminal inflorescence, on one of the upper 'bald suckers', or on one of the lower leafy laterals. More than one bud was often treated and bagged on a single inflorescence but in most cases such a group of buds were all of approximately the same length.

To obviate the possibility that grosser errors of technique could affect the results of the various treatments, the corollas of treated flowers were carried along with the seed bags and are, in most cases, still available for examination. If, in castration, an anther is overlooked the fact is at once noted when the withered corollas are examined both at the time of cleaning and at the time of sowing the seed produced. In only one case among the 800 treatments were anthers overlooked and a considerable amount of viable seed resulted. This check upon the results of castration and mutilation experiments is, to my mind, imperative. Willingness to discard all treatments concerning which there is the slightest doubt as to purity, is the only security obtainable in the finer details of performing the emasculations or mutilations. Despite the sterilization of instruments a bud was discarded if the forceps touched the stigma during any operation. Similarly, though only the middle one-third of the corolla tube was opened to admit the forceps, the too-close proximity of open flowers on neighboring inflorescences at the moment of treatment caused the discarding of the bud being operated upon. In addition to the 800 castrations and mutilations considered unimpeachable, 150 doubtful treatments were thrown away. The results, in the case of a few doubtful treatments which were saved, showed the necessity of such rigid discrimination.

Following the castration or mutilation of these 800 buds, there were 112 bags, involving nearly 200 flowers, in which one or more fruits developed to normal size. These fruits contained matured seed all of which was normal in appearance, though the majority was small in size, and a small proportion of which was normal in every way including the presence of endosperm and embryos. That there is a ready and frequent parthenocarpy, taking this term simply to indicate the production of normal fruits without pollination, in '*Nic. tabaccum* Cuba' there can be no doubt. This is the more remarkable since other Nicotiana species and varieties uniformly exhibit an early dropping of those flowers in which emasculation is not closely followed by pollination. It may be noted that, of the castrations and mutilations which were followed by the production of normal fruits, 75% were on the terminal inflorescence and the four or five 'bald suckers,' normal to this variety of *N. Tabacum*, and 25% only were on the lower leafy laterals. Similarly, buds treated within two weeks after the opening of the first flower on the plant matured fruits in 65 instances as compared with 57 instances in the case of treatments performed after that period. Certain plants were not treated until they had passed their first and heaviest flowering period. Such plants gave no significant increase in the proportion of fruits matured after treatment. The number of fruits matured from treated buds under 35 mm. in length was almost identical with the number produced from treated buds more than 35 mm. long. Almost equal numbers of four-parted and five-parted flowers produced fruits following treatment. However, seed containing endosperm and embryos or endosperm only seems to have resulted, in nearly all cases, from the treatment of the more normally formed, five-parted flowers. Similarly, the use of a single bud on an inflorescence for treatment seems, as was perhaps to be expected, to have been more efficient for the production of normal seed than the castration of more than one. Of the total 800 treatments less than one-third involved the pinching off of the stigma in addition to emasculation. Approximately one-tenth of the total number involved the pinching off of the stigma only. Nine treated flowers produced a little viable seed and, of these nine flowers, three represent treatments in which the stigma was removed.

The great majority of the seed produced in the parthenocarpic fruits was normal in appearance and well filled out but these seeds were smaller than the self-pollinated seeds and consisted of empty seed coats only. Samples of seed from each seed packet were bleached in strong Eau de Javelle and examined under magnification. Controls were em-

ployed in which self-fertilized seed of '*Nic. tabaccum* Cuba' was similarly treated. Three types of seed were isolated, according to this method, from the parthenocarpic fruits. First, seed which consisted of nothing but empty seed-coats normal in structure and completely filled out. Second, seed of the first type which contained traces of endosperm but no embryos.[9] Such endosperm tissue occurred as a sheath of cells lining the seed-coat or as a mass of cells at one end of the seed, the contents of which was rich in starch. These first two types were small seeds. The third type of seed isolated was identical with the self-pollinated seed both in size and in the possession of normally developed endosperm and full sized embryos. Seeds of type one were overwhelmingly in the majority in every case and the total number of seeds shown to be in every way normally matured, either by examination under the microscope or by the fact that they produced normal seedlings, was approximately thirty-five. They were produced, along with empty seeds and seed containing endosperm only, in approximately equal numbers in the nine parthenocarpic fruits. Four seedlings are maturing normally and are of fair size from nine seeds which were germinated out of a total of eighteen seeds available for germination. The eighteen seeds represent the proportion of the fifty undoubtedly viable seeds not used in the bleaching test.

We may conclude, then, that parthenocarpy is of frequent occurrence in '*Nic. tabaccum* Cuba' and that parthenogenesis, employing the term to mean the production of viable seed without pollination (cf. Winkler[8]), is also peculiar to this variety of *N. Tabacum*. With these unusual phenomena manifesting themselves during this first year of cultivation in our cultures I feel that there is a possibility, at least, that after further cultivation here parthenogenesis and parthenocarpy may become more nearly equal in the frequency of their occurrence. The experimental results above described should furnish a partial confirmation of those reported by Mrs. Thomas for her experiments on '*Nic. tabaccum* Cuba.' Nothing that has been said, however, must suggest that I desire to confirm her general results on the basis of which she concludes that parthenogenesis is peculiar to *Nicotiana* species in general. It must, on the contrary, be emphasized that our experiments, with all other species and varieties of tobacco, and those of a number of other investigators, point to exactly the reverse conditions and that we have no reason to suppose that parthenogenesis has ever before occurred in our cultures. I have no suggestion to offer, at the present time, as to the possible origin of this *Nicotiana tabacum* variety which exhibits

such marked divergence from the restricted method of fruit and seed production peculiar to other varieties of this species and to all other species of tobacco as far as known.

The experiments reported upon above have, in part, been made possible by an allotment from that portion of the Adams Fund of the United States Department of Agriculture granted to the Agricultural Experiment Station of the University of California. A more detailed report will appear in the University of California Publications in Botany, Volume 5.

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EXOGAMY AND THE CLASSIFICATORY SYSTEM OF RELATIONSHIP

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Lewis H. Morgan, in his *Systems of Consanguinity and Affinity* (Washington, 1871), established the fact that in a large part of North America, in India, in Africa, and in Oceania the natives use terms of relationship that designate not individuals but groups of individuals, and accordingly he labeled these systems as 'classificatory.' Later E. B. Tylor and others advanced the view that the classificatory system and exogamy—the rule that a person must marry outside of his own social group (clan or gens)—were merely two aspects of a single institution: that, in other words, primitive man classed together individuals belonging to the same exogamous division and separated individuals of different divisions. Quite recently this view has been advocated by W. H. R. Rivers. In his *Kinship and Social Organisation* (London, 1914) he correlates the classificatory system with exogamy, our own system with the family in the narrower sense of the term, and the descriptive system of, say, the Nilotic Negroes (in which a few primary terms designate the basic relationships and serve by their combination to describe all other relatives) with the patriarchal or extended family. The correlation of

the classificatory nomenclature with exogamy now requires empirical verification in the several areas of the globe, and the following is an attempt to make this test for North America.

Before undertaking this inquiry, however, the concept 'classificatory' must be supplanted by one that more adequately represents the phenomenon under discussion. For any particular 'classificatory' system is not moulded by a single factor but by a series of factors, and these are developed in varying degree in different systems. Hence the test must be applied to that common element which, consciously or unconsciously, differentiated the primitive terminologies in question from those current among ourselves in the minds of investigators. An examination of Morgan's earliest expressions on the subject indicates that it was the merging of lineal and collateral relatives—the use of a single term, e.g., for mother and mother's sister, for father and father's brother—that impressed this pioneer investigator, and this is the feature that actually characterizes the classificatory systems of all the regions of the globe. Our query is thus reduced to this form: Is the confusion of collateral with lineal relatives a function of exogamy?

The first question is, of course, how the exogamous tribes compare with those 'loosely organized,' i.e., those lacking exogamous divisions. One of the principal exogamous areas of North America is found in the United States east of the Mississippi. Practically throughout this immense territory the custom of exogamy is associated with a terminology that fails to distinguish collateral and lineal relatives. Among the Northwest Coast tribes of Canada the same association holds, and this applies likewise to those of the Plains tribes that possess a clan or gentile organization. The one doubtful exogamous region is the Southwest, for which we have practically no data except from the Tewa, where the correlation does *not* hold. An inquiry into the as yet unknown systems of the Keresan, Hopi, and Zũni pueblos is of the highest theoretical importance. When we turn to the loosely organized tribes we meet again with one exceptional region, that of the Mackenzie River, and several sporadic cases outside, where non-exogamous tribes are reported to possess a kinship terminology that is *ex hypothesi* to be expected together with exogamy. On the whole, however, the agreement with the Tylor-Rivers theory is highly satisfactory. The Eskimo, the Plateau Indians, the Californian tribes are loosely organized; and all of them tend to emphasize the distinction of such relatives as father and paternal uncle, mother and maternal aunt. It is important to note that these terminologies by no means resemble those of European languages. Among the Shoshonean tribes and the Kootenai, for example, relatives distinguished

in English are classed together through the extensive use of reciprocal terms, members of a related pair addressing each other by a common term. But these systems are 'classificatory' only in an etymological sense of the word, the basis of the classification being wholly distinct from that which moulds the collateral-lineal terminology of the systems customarily designated as 'classificatory.'

Summing up, we may say that there is a very high degree of correlation between the practice of exogamy and the ranging in a single category of collateral and lineal kin. The aberrant cases are relatively few and some of them are readily intelligible as the result of cultural influences from neighboring tribes. It is, of course, possible that the correlation may ultimately turn out to have an unexpected meaning, for example, that the emphasis on exogamy is misplaced and should be on definite organizations of any kind, whether exogamous or not. Such an interpretation might perhaps eliminate some of the at present anomalous instances of non-exogamous tribes sharing the nomenclature of exogamous peoples. At all events, the North American data furnish strong evidence in support of the Tylor-Rivers theory.

That the classification of kin by certain tribes is a function of the exogamous grouping, may be demonstrated in more rigorous fashion. Within the Siouan family there are tribes with kinship systems that not only fail to distinguish between collateral and lineal relatives but also class together members of distinct generations, which is contrary to the usual form of 'classificatory' nomenclature. Rivers has suggested exceptional forms of marriage to account for the exceptional mode of classification; as a matter of fact it may be shown that the apparent exceptions are merely the result of an unusually consistent application of the exogamous principle of grouping.

The following are the facts. The Crow and Hidatsa, Siouan tribes tracing descent through the mother, class the father's sister's son with the father; the father's sister's daughter, father's sister's daughter's daughter and all her female descendants through females with the father's sister. It is to be proved that these classifications are connected with the exogamous social grouping.

The facts in the case may be summed up by saying that a single term is applied to male members of the father's clan regardless of generation, and a single term to female members of the father's clan who belong to his own and all descending generations. If this objective statement also represents the psychological basis of the grouping, the terminology should be modified as soon as we pass outside the clan. We pass outside the clan when we take not the father's sister's daughter's daughter, but

the father's sister's *son's* daughter, since with exogamy and maternal descent she cannot belong to the father's clan. And actually we find that among the Hidatsa and Crow this relative is no longer classed with the father's sister but with the sister, this latter relationship following from the fact that *her* father is classed with *my* father. But between the status of a father's sister's daughter's daughter and a father's sister's son's daughter there is no ascertainable difference except that of clan affiliation. Therefore the terminological classification is a function of the exogamous group, which was to be proved.

The same conclusion may be established by eliminating the hypothetical factor in another way. The Siouan family embraces a number of tribes with patrilineal descent, of which the Omaha are the best known. In such tribes the father's sister's descendants are no longer, as among the Crow or Hidatsa, members of the father's exogamous group; and we find, as a matter of fact, that her son and daughter are classed not with her but with the sister's son and daughter. With paternal descent my father's sister is my group sister, and while the Omaha have a distinct term for the father's sister it seems that in some ways she is still regarded as a sister—both as regards her children and as regards her husband, who is classed with the brother-in-law. On the other hand, the mother's brother's son, mother's brother's son's son, and so forth, are all members of the same exogamous group if there is paternal descent, and the Omaha actually designate them by a single term. And again, as soon as we pass out of the exogamous group, the terminology varies: my mother's brother's *daughter's* son is my brother, not my mother's brother, since he no longer belongs to my mother's gens but is related to me solely through his mother, who is my 'mother' because *she* does belong to my mother's gens.

In short, passing from tribes with matrilineal to tribes with patrilineal descent within the Siouan stock, we find precisely those differences that logically follow from the assumption that the exogamous group lies at the basis of kinship classification; and passing within a particular tribe from relatives within the same group to relatives of otherwise similar status outside the group we at once find a difference in nomenclature. Hence the exogamous factor must have been a real cause in moulding the kinship terminology of at least some so-called classificatory systems.

SOLUTION OF AN INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS OF THE ANALYTIC TYPE

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Consider the infinite system of differential equations*

$$\frac{dx_i}{dt} = f_i(t; x_1, x_2, \cdots) = a_i + f_i^{(1)} + f_i^{(2)} + \cdots \quad (i = 1, 2, \cdots), \tag{1}$$

where a_i is a constant and $f_i^{(j)}$ is the totality of terms of f_i which are homogeneous in t, x_1, x_2, ... of degree j. That is, $f_i^{(1)}$ is a linear function of the infinitely many variables t, x_1, x_2, ...; $f_i^{(2)}$ is a quadratic function of the same variables; and so on. From the analogy with analytic functions of a finite number of variables, f_i will be said to be of the analytic type.

It is assumed that the following hypotheses are satisfied:

(H_1). $x_i = 0$ $(i = 1, 2, \ldots)$ *at* $t = 0$.

(H_2). *Finite real positive constants* $c_0, c_1, c_2, \ldots; r_0, r_1, r_2, \ldots, A$ *and* a *exist such that*

$$s = c_0 t + c_1 x_1 + c_2 x_2 + \ldots \tag{2}$$

converges if

$$|t| \leqq r_0, \qquad |x_i| \leqq r_i \qquad (i = 1, 2, \ldots) \tag{3}$$

and such that $Ar_i s^i$ *dominates* $f_i^{(j)}$ *and* $|a_i| \leqq Ar_i a$.

Since the series (2) converges if the relations (3) are satisfied, a finite constant M exists such that

$$S = c_0 \frac{r_0}{M} + c_1 \frac{r_1}{M} + c_2 \frac{r_2}{M} + \cdots < 1. \tag{4}$$

Consequently if $M|t| \leqq r_0$ and $M|x_i| \leqq r_i$ are satisfied, then

$$|f_i| \leqq A r_i \{a + S + S^2 + \cdots\} = A r_i \left\{ a + \frac{S}{1-S} \right\}. \tag{5}$$

* The problem of an infinite system of differential equations was first treated by E. H. Moore in his paper read at the Fourth International Congress of Mathematicians, held at Rome in 1908. His treatment was in the sense of General Analysis (cf. *Introduction to a Form of General Analysis*, New Haven Mathematical Colloquium, 1906), in which a general variable p is used in place of the index i of this paper having the range of positive integers, and his functions were not limited to those of the analytic type. The solution was made to depend upon the solution of an integral equation, in general non-linear, of the Volterra type. Simplifications and extensions of this theory were presented before the National Academy of Sciences, at Chicago, December 9, 1914.

Therefore under these limitations not only do the terms of each degree in the right member of equations (1) converge, but the whole right members converge.

The particular form of (H_2) was chosen so as to secure by one analysis as wide a range as possible of permissible values of $x_1, x_2, \ldots$. For example, if infinitely many of the c_i are bounded from zero the x_i must tend to zero for $i = \infty$. On the other hand, if Σc_i converges all the conditions so far imposed can be satisfied by x_i which are bounded from zero. In the latter case Σf_i does not converge.

If an analytic solution of (1) satisfying the initial conditions (H_1) exists, it will have the form

$$x_i = A_i^{(1)} t + A_i^{(2)} t^2 + \ldots \qquad (i = 1, 2, \ldots). \tag{6}$$

On substituting these series in (1) and equating coefficients of corresponding powers of t, it is found that

$$\begin{aligned} a_i^{(1)} &= a_i \qquad (i = 1, 2, \cdots), \\ 2a_i^{(2)} &= \frac{\partial f_i^{(1)}}{\partial t} + \sum_{j=1}^{\infty} \frac{\partial f_i^{(1)}}{\partial x_j} a_j^{(1)}, \\ &\cdots\cdots\cdots\cdots, \\ n a_i^{(n)} &= P_i^{(n)}(a_j^{(1)}, \cdots, a_j^{(n-1)}) \qquad (j = 1, 2, \cdots), \end{aligned} \tag{7}$$

where $P_i^{(n)}$ is a polynomial in $a_j^{(1)}, \ldots, a_j^{(n-1)}$ whose coefficients are linear functions of the coefficients of f_i with positive numerical multipliers. Hence the formal analytic solution of (1) is unique.

In order to prove the convergence of the series (6) for values of t whose moduli are sufficiently small, consider the solution of

$$\frac{d\xi_i}{dt} = A r_i \left\{ a + \frac{\sigma}{1 - \sigma} \right\} \qquad (i = 1, 2, \cdots), \tag{8}$$

where

$$\sigma = c_0 t + c_1 \xi_1 + c_2 \xi_2 + \cdots. \tag{9}$$

The right members of (8) dominate the respective right members of (1).

The formal analytic solution of (8) is

$$\xi_i = \alpha_i^{(1)} t + \alpha_i^{(2)} t^2 + \cdots \qquad (i = 1, 2, \cdots). \tag{10}$$

The coefficients of these series can be obtained by equations analogous to (7). They are therefore real and positive, and it follows from the fact that the right members of (8) dominate the right members of (1) that

$$\alpha_i^{(j)} \geqq |a_i^{(j)}| \qquad (i, j = 1, 2, \cdots). \tag{11}$$

Therefore if (10) converge for $|t| \leqq \rho$, then (6) also converge for at least the same values of t.

It follows from (8) that

$$\frac{1}{r_1}\frac{d\xi_1}{dt} = \frac{1}{r_2}\frac{d\xi_2}{dt} = \cdots = \frac{d\xi}{dt}.$$

The initial values of $\xi_1, \xi_2, \ldots$ are zero; hence on taking $\xi(0) = 0$, it follows that

$$\xi_i = r_i \xi \qquad (i = 1, 2, \cdots). \tag{12}$$

Therefore each of equations (8) reduces to

$$\frac{d\xi}{dt} = A\left\{a + \frac{c_0 t + C\xi}{1 - c_0 t - C\xi}\right\}, \tag{13}$$

where

$$C = c_1 r_1 + c_2 r_2 + \cdots, \tag{14}$$

which is a finite constant by (H_2).

It follows from the ordinary theory for a finite number of differential equations that (13) has an analytic solution which converges if $|t|$ is sufficiently small. Therefore equations (10) and (6) converge for at least the same values of t.

In general the limitations placed on t in order that the solution of (13) shall be known to converge are so restrictive that the corresponding x_i do not attain the boundary of the region for which the right members of (1) converge. The question arises whether the solution can be continued beyond its original domain.

Suppose equations (6) converge for $t = t_0$ and let the corresponding value of x_i be $x_i^{(0)}$. Suppose

$$c_0|t_0| + c_1|x_1^{(0)}| + c_2|x_2^{(0)}| + \ldots = S_0 < S_1 < 1.$$

Then let

$$x_i = x_i^{(0)} + y_i, \qquad t = t_0 + \tau. \tag{15}$$

The differential equations (1) become in the new variables

$$\frac{dy_i}{d\tau} = b_i + g_i^{(1)} + g_i^{(2)} + \cdots \qquad (i = 1, 2, \cdots), \tag{16}$$

where $g_i^{(j)}$ is the totality of terms in the ith equation which are homogeneous in $\tau, y_1, y_2, \ldots$ of degree j.

It follows from (1) that the explicit expressions for b_i, $g_i^{(1)}$, $g_i^{(2)}$, ... are

$$\begin{aligned} b_i &= f_i(t_0; x_1^{(0)}, x_2^{(0)}, \cdots), \\ g_i^{(1)} &= \frac{\partial f_i}{\partial t}\tau + \frac{\partial f_i}{\partial x_1}y_1 + \frac{\partial f_i}{\partial x_2}y_2 + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ g_i^{(n)} &= \frac{1}{n!}\left[\frac{\partial f_i}{\partial t}\tau + \frac{\partial f_i}{\partial x_1}y_1 + \frac{\partial f_i}{\partial x_2}y_2 + \cdots\right]^n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned} \tag{17}$$

where the power indicated in the last equation is symbolic such that

$$\left(\frac{\partial f_i}{\partial t}\tau\right)^{n_0}\left(\frac{\partial f_i}{\partial x_1}y_1\right)^{n_1}\left(\frac{\partial f_i}{\partial x_2}y_2\right)^{n_2}\cdots = \frac{\partial^n f_i}{\partial t^{n_0}\partial x_1^{n_1}\partial x_2^{n_2}\cdots}\tau^{n_0}y_1^{n_1}y_2^{n_2}\cdots$$
$$(n_0 + n_1 + \cdots = n). \tag{18}$$

After the partial derivatives have been formed, t, x_1, x_2, ... are replaced by t_0, $x_1^{(0)}$, $x_2^{(0)}$, ... respectively.

Since the transformation (15) is linear, $g_i^{(n)}$ depends only on $f_i^{(n)}, f_i^{(n+1)}, f_i^{(n+2)}, \cdots$. It follows from the fact that $f_i^{(n)}, f_i^{(n+1)}, \cdots$ are dominated by $Ar_iS^n, Ar_iS^{n+1}, \cdots$ respectively that

$$\left|\frac{\partial^n f_i}{\partial t^{n_0}\partial x_1^{n_1}\partial x_2^{n_2}}\right|_{\substack{t=|t_0| \\ x_j=|x_j^{(0)}|}} \leqq Ar_i\left[\frac{\partial^n\left(\frac{S^n}{1-S}\right)}{\partial t^{n_0}\partial x_1^{n_1}\partial x_2^{n_2}\cdots}\right]_{\substack{t=|t_0| \\ x_j=|x_j^{(0)}|}}.$$

It is easily found that

$$\left[\frac{\partial^n\left(\frac{S^n}{1-S}\right)}{\partial t^{n_0}\partial x_1^{n_1}\partial x_2^{n_2}\cdots}\right]_{\substack{t=|t_0| \\ x_j=|x_j^{(0)}|}} = \frac{n!\,c_0^{n_0}\,c_1^{n_1}\,c_2^{n_2}\cdots}{(1-S_0)^{n+1}}. \tag{19}$$

Now let

$$c_0^{(1)} = \frac{c_0}{1-S_0}, \qquad c_1^{(1)} = \frac{c_1}{1-S_0}, \qquad c_2^{(1)} = \frac{c_2}{1-S_0} \tag{20}$$

The series

$$T = c_0^{(1)}\tau + c_1^{(1)}y_1 + c_2^{(1)}y_2 + \cdots$$

converges if

$$|\tau| \leqq r_0(1-S_0), \qquad |y_1| \leqq r_1(1-S_0), \qquad |y_2| \leqq r_2(1-S_0), \cdots,$$

and $|T| < 1$ if

$$|\tau| \leqq \frac{r_0(1-S_0)}{M}, \qquad |y_i| \leqq \frac{r_i(1-S_0)}{M}.$$

Moreover $g_i^{(n)}$ is dominated by $\frac{Ar_i}{1-S_0} T^n$. Therefore equations (16) have the essential properties assumed to hold for (1), and their solution converges for $|\tau|$ sufficiently small.

It is easy to imagine a physical problem which satisfies the conditions of this theory. For example, suppose the number of mutually gravitating bodies in the universe whose masses are bounded from zero is infinite. If beyond a finite number of them (which may be arbitrarily great) their initial distances from one another increase, as the number of bodies increases, with sufficient rapidity, it is easy to show that all the hypotheses are satisfied. In this case there is a rigorous, though limited, solution of the problem of infinitely many bodies moving subject to their mutual attractions.

SEX RATIO IN PIGEONS, TOGETHER WITH OBSERVATIONS ON THE LAYING, INCUBATION AND HATCHING OF THE EGGS

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The conclusions[1] here presented are the results of a study of the records which have accumulated from 1907 to 1914 in connection with investigations of inheritance in pigeons begun at the Rhode Island Agricultural Experiment Station and later continued at the Experiment Station at Madison, Wisconsin. It is impossible in a brief space to present the data upon which the conclusions are based; for these the reader is referred to the complete report. Furthermore, although the conclusions are here presented somewhat dogmatically and as if of general application, and while we believe that they will probably be found in the main to apply generally to domestic pigeons, they are nevertheless based almost entirely on the data of the experiments mentioned and there is, therefore, no positive assurance that the results would be the same with other stock or under different conditions. The number of data obtained were, however, very considerable for pigeons, and it is felt they accordingly furnish a good foundation for the conclusions drawn.

It is commonly believed by pigeon breeders and others that from the two eggs of a clutch a pair of offspring, that is a male and a female, are produced either invariably, or at least in a great majority of instances. Furthermore, it is maintained that of this pair the male hatches from the egg which is laid first, while the egg laid later produces the female. The

breeder has also found it difficult to explain the apparent excess of males which he commonly finds in the birds saved for breeding stock. The results here presented throw conclusive light on these and on other similar questions.

Sex ratio.—The normal ratio of the sexes of pigeons hatched is 105 males to 100 females.

Death rate.—The death rate of squabs is especially high for the first two or three days after hatching, and at about 10 to 15 days of age.

Differential mortality.—(1) When the two squabs from the same clutch of eggs are of distinctly different size before the banding age (10 to 15 days) the larger squab is more often a male than a female. (2) The death rate for the two sexes in bisexual broods is essentially equal. (3) There is no marked tendency for one sex to be weaker than the other in bisexual broods, and there is only a slight indication that more males than females from such broods survive to adult life—placed at 6 months. (4) A consideration of the ratio of males to females in each of the age groups defined[2] does not indicate a high relative mortality of females in the ages preceding the adult stage. (5) There is a high mortality of both sexes during the first two or three years of their adult life, and this is especially high in the females between the ages of one and two years. (6) The higher mortality of females at early adult ages, together with the higher proportion of males hatched (105: 100), may be in large part responsible for the prevailing notion of a considerable excess of male pigeons in adult populations and seems to furnish real substantiation for this notion. The fact that males are generally more easily recognized than females probably adds to this impression.

Ratio of bisexual to unisexual broods.—(1) The number of unisexual broods, in which the squabs are either both male or both female, somewhat exceeds in our records the bisexual broods (one squab of each sex), but the odds against the numbers obtained representing a potential equality are very slight. These facts are directly contradictory to the commonly accepted statement that the two eggs usually produce a male and a female squab. (2) Considering only the unisexual broods, the number of 'both males' to 'both females' is practically equal.

Sex with respect to order of laying.—A comparison of the numbers of each sex hatched from first eggs and from second eggs respectively, shows no tendency for the former to produce exclusively males and the latter females but, as a matter of fact, more males than females are hatched from both.

Time of laying.—(1) The mean time of laying of the first egg is about 5 p.m., and of the second egg about one o'clock of the afternoon of the

second day following. (2) The mean interval between the laying of the two eggs is practically 44 hours. (3) The mean time between the laying of the first and second eggs decreases progressively in the months from February to July inclusive. (4) There is a very sensible positive correlation (0.38 ± 0.057) between the time of laying of the first and of the second egg. The equation of the regression straight line is $y = -0.48 + 0.378x$, where x is the time p.m. of laying of the first egg, and y that of the second. This enables one to calculate the most probable time of laying of the second egg when the time the first was laid is known.

Time of hatching.—(1) The mean time of hatching of the first egg is 16.5 days after the laying of the second. (2) The mean time of hatching of the second egg is 17 days after it is laid. (3) On the average, therefore, the time from laying to hatching of the first egg is nearly a day and a half longer than it is for the second egg. This is probably to be accounted for by the fact that the first egg receives very little incubation until the second is laid. (4) There is a high correlation between the times of hatching of the two eggs of a clutch.

Control of sex.—So far as the data obtained go, they appear to indicate that sex in pigeons is determined according to the laws of chance.

Time of sitting.—(1) In case the eggs do not hatch they are seldom abandoned at the end of the normal period of incubation, but the birds continue to sit on them for a time longer. (2) The length of time they will continue to incubate the eggs varies, but averages practically six days beyond the normal period, making the mean total time of incubation, when the eggs do not hatch, 23 days after the laying of the second egg. (3) This continuance of incubation beyond the normal time under such circumstances constitutes a 'factor of safety' in the incubating instinct. (4) A pair of Mourning Doves continued to sit on substituted eggs for four days after their own had hatched in an incubator, thus disproving Raspial's assertion that wild birds have an 'exact notion of the time required for the eggs to hatch.'

[1] Summary of the results of a report bearing the same title and published as *Bulletin No. 162 of the Rhode Island Agricultural Experiment Station.*

[2] GROUP A. *Hatching period.* Embryos which were fully developed but which failed to hatch ('dead in shell') and birds which died within five days after hatching.

GROUP B. *Nestling period.* Squabs dying at from 6 to 28 days, inclusive.

GROUP C. *Immature or juvenile period.* Squabs living over 28 days but dying before six months of age.

GROUP D. *Adult period.* All birds living to the age of six months or more.

VIVIDIFFUSION EXPERIMENTS ON THE AMMONIA OF THE CIRCULATING BLOOD

By Alice Rohde

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Presented to the Academy, May 5, 1915

The fact that the ammonia content of shed blood under aseptic conditions increases is well known and is taken into account as much as possible by a rapid procedure in methods for the determination of preformed ammonia. Suggestions as to the source of this slowly liberated ammonia have as yet led to no positive findings. My experiments were undertaken to determine whether with aseptic measures the formation of ammonia occurs in diffusible constituents of the blood after their separation from the non-diffusible constituents according to the vividiffusion method of John J. Abel[1] and his collaborators. The dialysate, obtained when the vividiffusion apparatus was attached to the femoral artery and femoral vein of a dog for periods varying from three to seven hours, was studied for the production of ammonia in excess of that present at the time of dialysis. The results were compared with those from shed blood under similar conditions. It was found that in a dialysate obtained from circulating blood there is no liberation of ammonia comparable to that which takes place under aseptic conditions in shed blood.

	Milligrams of NH_3-N *per 100 cc.*
Blood	
Sample taken at close of dialysis	0.30
Sample taken at close of dialysis and allowed to stand 36 hours	0.63
Dialysate	
Sample after 5½ hours dialysis	0.27
Sample after 7 hours dialysis	0.29
Sample removed after 7 hours dialysis and allowed to stand 36 hours	0.28

The ammonia content of the blood is doubled in twenty-four hours while that of the dialysate shows no increase. An equilibrium between the ammonia content of the dialysate and of a sample of the circulating blood was reached.

The slowly evolved ammonia has its source therefore in non-diffusible constituents of the blood.

Details of the methods with additional tables appear in the June (1915) number of the *Journal of Biological Chemistry*.

[1] On the removal of diffusible substances from the circulating blood of living animals. *J. Pharmacol. Exp. Therap.*, 5, 275 and 611 (1914).

126 PARABOLIC ORBITS OF METEOR STREAMS

By Charles P. Olivier

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Presented to the Academy, May 10, 1915

The study of meteors as astronomical bodies began in the early part of the last century. The occurrence of the great showers of 1833 and 1866 showed the importance of the meteoric astronomy. When it was proved, soon after the latter appearance, that this meteor stream and Tempel's Comet of 1866 had the same orbit about the sun, the interest in meteors increased enormously. Though millions of meteors are swept up by the earth every twenty-four hours, still these little bodies also obey the law of gravitation, and, in obedience to this law, meteors pursue their orbits about the sun.

Astronomy is interested in proving the connection between comets and meteors, in investigating the distribution of meteoric matter in space with special regard to that near the earth's orbit, in finding the cause of the zodiacal light, and the action of meteoric matter as a resisting medium and as an absorber of light in space. Meteorology is interested in the study of meteor trains and meteor heights, because the former give us our only information about the velocity and directions of wind currents in the upper strata of our atmosphere, the latter certain information as to how high this atmosphere must extend.

To arouse interest in this subject the American Meteor Society was organized in 1911, and the results mentioned in this paper were worked out from the 2800 observations contributed by its members. These results appear in full in Volume 2, Part 4, of the Publications of the Leander McCormick Observatory of the University of Virginia. While the most important contribution to science was the 126 parabolic orbits of meteor streams, the most interesting result, undoubtedly, was the final proof of the connection of Halley's Comet and the η Aquarid meteors. These bodies were shown to follow orbits similar to Halley's Comet and present one of the best instances of the slow disintegration of such a body. The question whether radiants are stationary was studied in considerable detail, and a negative conclusion was reached, principally because motion was detected in the radiant point of the Orionid meteors, which are held to be the typical case by certain authorities. The tabular results of the paper are of interest to anyone who has even touched on the subject, showing as they do at a glance just what one can expect from a few hours' observation at favorable times of the year. The National Academy of Sciences at its annual meeting has granted

a small sum from the J. Lawrence Smith fund for the purpose of extending the work.

For observations of meteors a technical knowledge of astronomy is not necessary. Anyone interested as an amateur is invited to write to the Leander McCormick Observatory. Maps and directions for observing will gladly be forwarded. It is earnestly hoped that a large number of amateurs will assist us in the extension of this work.

THE BASAL SILURIAN FORMATIONS OF EASTERN NORTH AMERICA

By Charles Schuchert

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Presented to the Academy, April 30, 1915

At the base of the Silurian system of rocks in the state of New York and in the Appalachian Mountains occurs the Medina sandstone. In tracing this formation from the gorge of the Niagara River northwestward into Ontario, the sandy phase of the Medina is seen to change gradually into a muddy one and finally into a limestone, as is the case on the Manitoulin Islands of Lake Huron. These early Silurian rocks of Ontario, known as the Cataract formation, have long been regarded as equivalent to the Clinton formation of New York, but are now known to hold the time of the Medina formation. In another direction, in Ohio, Indiana, and Kentucky, the basal Silurian strata, the Brassfield formation, are also seen to be the equivalent of the Cataract formation of the north.

The faunas of these three marine formations are quite different. The Medina sandstone has the smallest assemblage, with 22 species, while the Cataract limestones have at least 76 forms, and the Brassfield limestones 140 kinds of invertebrates. There are but 7 species in common between the Medina and Cataract, while the latter has 24 forms repeated in the Brassfield.

Each one of the three formations represents a different marine basin. The Medina is of the Appalachian province, is a sandstone formation, and finally invades to a slight extent the area of the Cataract. The Brassfield province lies in the main west of the Cincinnati axis, is of southern origin, with limestone-making seas, spreads also up the southern portion of the Appalachian province, and finally likewise invades slightly the area of the Cataract sea. On the other hand, the Cataract province spreads westward through the Saint Lawrence embayment, and finally, in eastern Ontario and northwestern Ohio, unites with the

other two provinces. The normal marine junction of the Cataract and Brassfield seas is prevented by the Medina delta of sands. For these reasons, Medina, Cataract, and Brassfield are to be retained as names for independent marine faunas and formations. The details leading to these conclusions are set forth in a contribution entitled 'Medina and Cataract formations of the Siluric of New York and Ontario,' *Bulletin of the Geological Society of America.*

A METHOD OF OBTAINING COMPLETE GERMINATION OF SEEDS IN *OENOTHERA* AND OF RECORDING THE RESIDUE OF STERILE SEED-LIKE STRUCTURES

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All genetical workers with *Oenothera* shortly become aware that generally only a small proportion of the seed-like structures sown in their seed pans produce seedlings before the pans are emptied to give place in the hot house for the developing culture. My own practice has been to keep seed pans from eight to ten weeks only, as it is uncertain, if sowings are made in January, whether seedlings appearing later can be brought to maturity during the summer. For the past three seasons I have counted the seeds sown and thus have obtained some information on the relative degrees of fertility in my cultures which in some cases have been surprisingly low. But this procedure does not give accurate data on the proportion of fertile seeds to sterile seed-like structures for the reason that germination of *Oenothera* seeds in earth is very irregular and may be delayed far beyond the time that it is convenient or possible to keep the seed pans. Seeds sown in earth are obviously lost for further enquiry as to the fact of their viability, a proportion of seedlings appears but, as for the residue, that cannot be examined.

The technical problem of obtaining from sowings of seeds cultures that will accurately represent the genetical possibilities of the sowing is under these conditions difficult. In *Oenothera* work this problem is vital for exact studies since through the delayed germination may be lost not only peculiar individuals but possibly, in the case of hybrids, entire classes of segregates. It is probably safe to say that no culture of *Oenothera* has as yet been described in which we may feel confident that all of the viable seeds have germinated. Consequently we cannot be certain that any of the reported percentages of 'mutants' or ratios of segregates from hybrids are correct. Furthermore it is important

to determine for oenotheras the ratio of fertile seeds to the structures that in outward appearance cannot be distinguished from seeds but which contain no living tissue or only a small amount of endosperm, with an embryo, if present at all, so small or abnormal that it cannot produce a seedling. We are indebted to O. Renner[1] for the first detailed investigation of the sterile seeds of an *Oenothera* and for the formulation of a most interesting hypothesis as the result of his studies on *O. Lamarckiana*.

During the past winter I have tested the seed fertility of fifty species, races, or hybrids of *Oenothera* and as a result of these experiments I have become convinced that genetical research in this group must in the future adopt methods that will ensure a rapid and complete germination of the viable seeds and at the same time conserve the sterile seeds or undeveloped seed-like structures for examination. Such methods will require that seeds be germinated in some convenient receptacle and subsequently be set in earth leaving as a residue within the receptacle all sterile structures.

The method employed this winter is briefly as follows: In a Petri dish $3\frac{3}{4}$ inches in diameter was placed a pad of circular filter papers 3 inches across and about $\frac{1}{4}$ inch thick. The dish, cover, and paper pad were then sterilized by heat after which boiled tap-water was added in such quantity that the thoroughly soaked paper pad lay in the center of the dish surrounded by water. The seed-like structures were then spread over the surface of the pad and the dish covered. The seeds, therefore, lay on a very wet surface and in a moist chamber; more water was added when evaporation materially lowered the level in the dish. The tap-water was boiled to avoid the introduction of algae; fungi appeared in some of the cultures but apparently found the seed coats an unfavorable substratum for they grew very little; bacteria were likewise not troublesome.

The Petri dishes with seeds were placed in shaded parts of the same hot house with cultures sown in earth and were thus under similar conditions as to temperature. The possible germination within the dishes was always very much more prompt than for a similar sowing of seeds in the earth and proved to be complete when sufficient time was allowed. The advantages of Petri dishes as receptacles are those of convenience in the removal of germinated seeds and in the collection and examination of the residue. Petri dishes are also readily stacked and upon labels over the covers may be recorded the data of the experiment.

The rate of germination in the Petri dishes under the conditions of

the hot house was irregular. Complete germination in some species required only three weeks, in other forms as much as from six to ten weeks, and in some cultures I found a few ungerminated seeds after three and one half months. In warm, sunny weather the house during part of the day was at 90°F., and these high temperatures were frequently followed by a burst of germinations. It is thus clear that the best results in experimental germination will be obtained when cultures are placed in chambers or incubators the temperature of which can be regulated with accuracy.

Hugo De Vries[2] has recently published a method of stimulating seed germination in *Oenothera* that seems likely to eliminate largely the irregularity and slowness of germination described above. He reports that well soaked seeds after being subjected to a pressure of 6–8 atmospheres for 1–3 days at room temperature will germinate in large percentages when removed to an oven at 30°C. His hypothesis for this interesting behavior assumes that the hard inner coats of the *Oenothera* seeds have narrow microscopic slits filled with air and that ordinarily water enters the seed with difficulty but under pressure can be forced to points where it may readily be absorbed. De Vries thus contributes the practical suggestion of an automobile pump and a strong chamber such as an autoclave into which receptacles containing the well soaked seeds may be packed. If complete germination is not obtained after one treatment to compressed air successive treatments alternating with normal atmospheric pressure and perhaps variation in temperature conditions may bring forward stubborn cultures.

The residue that is left after germination is believed to be complete has in my experience generally ranged from structures fully as large as the larger of the viable seeds to structures smaller than seeds, which in turn grade into the remains of numerous unfertilized, or abortive ovules, represented in the seed capsules by a light brown powder. It is an easy matter to determine whether or not the structures in the residue, seed-like in size and appearance, contain an embryo. The structures may be opened with the point of a needle or scalpel, or they may be squeezed flat between the tips of a strong pair of forceps. The presence of an embryo is at once made evident and the germ may be readily examined. Generally the seed-like structures will be found to be entirely empty or to contain only a trace of soft tissue. Structures smaller in size than normal seeds can be examined microscopically. The residue may be conveniently preserved as a record by arranging the structures on paper and covering them with a solution of shellac in alcohol, or with a solution of glue.

To illustrate the advantages of germinating *Oenothera* seeds under experimental conditions over the old practice of sowing upon earth I will give the results for a problem as attacked under the old and new method. In 1914 I germinated on earth seeds from an F_1 hybrid plant, 13.35ac, of the cross *O. franciscana* × *O. biennis*. A sowing of 819 seed-like structures produced a culture of 402 seedlings, a germination of about 50%. The culture was grown partly to test the inheritance of a character (red coloration of the papillae on the stems and ovaries) present in *franciscana* and absent in *biennis*. This character was fully dominant in the F_1 of this cross and in its reciprocal. It seemed reasonable to expect that a proportion of the plants in the F_2 generation would present the clear green stems and ovaries (recessive) of *biennis*, but I found no plants of this type in the culture of 1914. This winter (1915) I germinated in Petri dishes seeds from the same F_1 hybrid plant, 13.35ac, and obtained 761 seedlings from about 921 seed-like structures,—a germination close to 82%. The new method had raised the proportion of germination more than 30%. Of the 761 seedlings I was able to bring 748 to rosettes, the 13 which died probably belonging to a small group of etiolated dwarfs difficult to grow. It becomes a matter of interest to see whether or not in this culture of 1915 a group of green-stemmed plants will appear.

[1] Renner, O., Befruchtung und Embryobildung bei *Oenothera Lamarckiana* und einigen verwandten Arten. *Flora, Jena*, 107, 115 (1914).

[2] De Vries Hugo, The Coefficient of Mutation in *Oenothera biennis* L. *Bot. Gaz., Chicago*, 59, 169 (1915).

THE OSMOTIC PRESSURE OF THE IONS AND OF THE UNDISSOCIATED MOLECULES OF SALTS IN AQUEOUS SOLUTION

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The concentration of the ions (C_i) and that of the undissociated molecules (C_u) are involved in the equation, $C_i^2 / C_u = K$, expressing the application of the law of mass-action to solutions of electrolytes. Hence the large divergence from this law which strong electrolytes exhibit may be due to the behavior of the ions, to that of the undissociated molecules, or to that of both. This so called anomalous behavior may be simply expressed by saying that for one or for both of these molecular species van't Hoff's law, $\Pi = CRT$, where Π is the osmotic pressure, C is the concentration in mols per unit-volume, R is the gas-

constant, and T is the absolute temperature, does not hold; for, if all of the molecular species entering into an equilibrium obey this law, the law of mass-action is a thermodynamic necessity.[1]

In this paper is outlined and applied a method for determining the extent to which the ions and the undissociated molecules deviate from van't Hoff's law, that is, for determining the relation between the osmotic pressure of the ions or of the undissociated molecules and their concentration. The significance of the results, particularly as applied to the calculation of the degree of ionization and to the validity of the law of mass-action, is discussed.

Besides the laws of thermodynamics, the assumptions involved in these calculations are that (1) in a dilute solution of a di-ionic electrolyte the osmotic pressure due to each of the two ions is the same, and that (2) the degree of ionization (γ) is given by the conductance-viscosity ratio ($\lambda\eta/\lambda_o\eta_o$), the concentration of each ion then being $C\gamma$ and that of the undissociated molecules $C(1-\gamma)$. These assumptions are generally accepted; they are, however, briefly discussed below.

The total osmotic pressure (Π) of a solution is due in part to that of each of the two ions and in part to that of the undissociated molecules. Since by the first assumption above stated, the osmotic pressure of one ion is equal to that of the other, the relation $\Pi = 2\Pi_i + \Pi_u$ follows. The principles of thermodynamics, together with the same assumption, lead for equilibrium in a solution of di-ionic electrolyte, to the relation[1]

$$2d\Pi_i/C_i - d\Pi_u/C_u = 0 \qquad (1)$$

If now C_i and C_u are calculated from conductance data, and if the total osmotic pressure of the solution be known (from direct measurements or from freezing-point determinations, etc.), there remain in these two equations but two unknown quantities Π_i and Π_u.

The solution of these two equations gives for the rate of change of the osmotic pressure of the ions with their concentration the relation:

$$\frac{d\Pi_i}{dC_i} = \tfrac{1}{2}\left[\frac{\Pi}{C} + \frac{\gamma}{2\cdot 303}\,\frac{d(\Pi/C_i)}{d\log C_i}\right] \qquad (2)$$

The first term Π/C may be calculated directly from the data. The second term, which is small in comparison with the other, may be conveniently evaluated by the graphic method of plotting values of Π/C_i against those of $\log C_i$ and determining the tangents. Having calculated $d\Pi_i/dC_i$ for a series of concentrations, values of Π_i/C_i may be determined by integration, either graphic or algebraic.

In order to determine the similar ratios for the undissociated molecules, the equation corresponding to (2) might be employed. In practice, however, it is more convenient to use a different method. By eliminating K, C_i and C_u from Storch's equation, $[C_i^n / C_u = K]$, from the differential equation obtained from this and from equation (1), the relation

$$d\Pi_u/dC_u = \frac{2}{n} d\Pi_i/dC_i$$

is obtained. That is, having calculated values of $d\Pi_i/dC_i$ and of n at any given concentration, by the methods which have been previously developed,[2] over a concentration-range, values of $d\Pi_u/dC_u$ may be determined, and hence by integration the osmotic pressure of the undissociated molecules may be calculated.

The results of such calculations for a few representative di-ionic electrolytes for whose solutions reliable freezing-point data[3] are available are given in the table. They are given as ratios of the form $\frac{d\Pi}{dC}/RT$. The deviations of these from unity show the degree of deviation from van't Hoff's law at the different concentrations. For most purposes the ratio $d\Pi/dC$ here considered is more useful than the integral ratio Π/C; for it is the former ratio which is introduced into thermodynamic relations in order to derive the mass-action law, the electromotive-force equation, etc. Further, the differential ratio gives information of a more definite character regarding the properties of the solution at any given concentration than does the integral ratio. The value of the former depends only upon the properties of the solution of the concentration in question, while that of the latter is influenced by, and to a great extent depends upon, the properties of the solutions of all concentrations from the infinitely dilute solution up to that in question.

Values of the Ratio $\frac{d\Pi_i}{dC_i} \Big/ RT$ *for the Ions*

Equivalents per liter	LiCl	NaCl	KCl	KNO_3	$CuSO_4$
0.001	.....	0.996	0.987	.....	0.927
0.002	.....	0.993	0.983	.....	0.919
0.005	.....	0.990	0.983	0.990	0.899
0.01	0.985	0.986	0.979	0.983	0.867
0.02	0.993	0.983	0.977	0.972	0.824
0.05	0.986	0.980	0.970	0.946	0.756
0.1	1.000	0.973	0.967	0.926	0.734
0.2	1.015	0.963	0.960	0.888	.....
0.3	1.046	0.965	0.958	0.850	.....
0.5	1.110	0.967	0.956	.....	.....

Values of the Ratio $\frac{d\Pi_u}{dC_u} \Big/ RT$ *for the Undissociated Molecules*

Equivalents per liter	LiCl	NaCl	KCl	KNO_3	$CuSO_4$
0.001	.....	1.263	1.249	.....	1.090
0.002	.....	1.280	1.262	.....	1.091
0.005	.....	1.305	1.289	1.279	1.085
0.01	1.334	1.323	1.305	1.281	1.067
0.02	1.366	1.343	1.325	1.275	1.046
0.05	1.390	1.375	1.350	1.246	1.025
0.1	1.439	1.394	1.375	1.217	1.055
0.2	1.500	1.413	1.396	1.159	.....
0.3	1.577	1.446	1.412	1.100	.....
0.5	1.728	1.488	1.432	.....	.....

The results in the table show that, even in solutions as dilute as 0.001 normal, neither the ions nor the undissociated molecules obey van't Hoff's law, though with decreasing concentration both become more nearly normal.

The osmotic pressure of the ions is less than that calculated from this law. This is true in every case except for solutions of lithium chloride at 0.1 normal and above; hydration is doubtless the cause of this exception.

It can be shown, moreover, that the behavior of the ions may be represented within the error of the freezing-point determinations by an equation of the form

$$\frac{d\Pi_i}{dC_i} = RT\,(1 + kC_i^n)$$

The results seem to justify the assumption that the osmotic pressures of the two ions in a solution of a di-ionic electrolyte are the same up to moderate concentrations. For example, the behavior of the chloride ion is within the experimental error the same whether calculated from the data for sodium chloride or from that for potassium chloride.

The undissociated molecules have an osmotic pressure considerably greater than that calculated from van't Hoff's law. At moderate concentrations the various salts do not behave similarly. In the case of the chlorides the deviations continue to increase with the concentration, but in that of the nitrates a maximum is reached about 0.02 normal. This somewhat surprising result, that the behavior of the least hydrated is the most complex, is connected with the fact that the exponent n is for these salts a minimum at about 0.02 normal.

For solutions more dilute than 0.01 normal and in some cases to a much higher concentration, an empirical equation of a form similar to that which holds for the ions expresses the relation between the osmotic

pressure of the undissociated molecules and their concentration. In the case of potassium chloride these relations, one for the ions and one for the undissociated molecules, express the results for solutions between 0.001 normal and 0.5 normal, so closely that the average difference between the total osmotic pressure of the solution calculated from them and that experimentally determined is but 0.10 percent.

The behavior of the undissociated molecules of copper sulphate is remarkable in two particulars. First they show an almost constant deviation from van't Hoff's law from 0.001 normal to 0.02 normal. In the second place, while the bivalent ions are considerably more abnormal than are the univalent ions, the undissociated molecules of this salt obey van't Hoff's law much more closely than do those of the uni-univalent salts.

H. Jahn[4] and recently G. N. Lewis[5] have considered it to be less reasonable to assume that (1) the concentration of the ions is proportional to the electrical conductivity of the solution, than it is to assume that (2) the concentration of the ions is proportional to their osmotic pressure or activity, that is, that the ions are normal or that they obey van't Hoff's law. Parallel to this latter assumption is the one that (3) the undissociated molecules are normal. The degree of ionization calculated upon any of these three assumptions is different from that calculated upon the other two. That is, the acceptance of one compels the rejection of the others.

It may be shown in another way that the experimental data are such that it is not possible for both the ions and the undissociated molecules of strong electrolytes to be normal. By combining the mass-action law relation, $C_i^2/C_u = K$, and the differential equation derived from this with the thermodynamic equation (1) above, it can be shown that the necessary and sufficient condition that the law of mass-action hold over a certain concentration range is that at all concentrations $d\Pi_i/dC_i = d\Pi_u/dC_u$; that is, that the rates of change of the osmotic pressures of the ions and of the undissociated molecules with their respective concentrations be the same. Since the mass-action law is not obeyed by solutions of strong electrolytes this relation does not hold, and the relation $\Pi = CRT$ cannot be true for both molecular species. The computation of the mol-number i from freezing-point data, and the calculation of the degree of ionization on this basis, assumes that both the ions and the undissociated molecules are normal. Since the resulting values do not obey the mass-action law, the method is inconsistent and theoretically incorrect. It has long been recognized that actual gases are not 'perfect' gases. Unfortunately the idea that in

general solutes may be treated as normal or 'perfect' solutes is much more persistent.

The magnitude of the deviation of the strong electrolyte from the normal behavior depends greatly upon the terms in which this deviation is expressed. The equilibrium expression C_i^2/C_u increases with the concentration. Thus for potassium chloride it increases from 0.026 at 0.001 normal to 0.55 at 0.1 normal, an increase of 21 fold. The results given in the table show that this behavior is accounted for by a deviation of 3.3% in the osmotic pressure of the ions and of 37% in that of the undissociated molecules. The percentage deviation when expressed as here in terms of osmotic pressure is smaller than when it is expressed, as Lewis[5] has done, in terms of activity; this follows from the logarithmic relation between the two.

A more thorough study of these topics is appearing in *The Journal of the American Chemical Society*.

[1] E. W. Washburn, *J. Amer. Chem. Soc.*, 32, 484–5 (1910).

[2] S. J. Bates, *J. Amer. Chem. Soc.*, 35, 519 (1913).

[3] The data have been taken in great measure from the compilation of A. A. Noyes and K. G. Falk, *J. Amer. Chem. Soc.*, 32, 1011 (1910). This has been supplemented by the recent measurements of F. Flügel, *Zs. physik. Chem.*, 79, 577 (1912), of W. A. Roth, *Ibid.*, 79, 599 (1912), of E. W. Washburn and D. A. McInnes, *J. Amer. Chem. Soc.*, 33, 1686 (1911) and of L. H. Adams, *Ibid.*, 37, 481 (1915).

[4] *Zs. physik. Chem.*, 33, 545 (1900).

[5] *J. Amer. Chem. Soc.*, 34, 1631 (1912).

THE EXTENSION OF THE SPECTRUM BEYOND THE SCHUMANN REGION

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The violet limit of the spectrum determined by direct eye observation lies in the neighborhood of 4000 angstrom units; with a glass prism and lenses the spectrum may be followed photographically to wave length 3000 or thereabout; with a quartz system or with a reflecting grating, the limit may be pushed to wave length 1850. Victor Schumann showed that the absorption of the air and of the gelatine of the photographic plate were responsible for the abrupt termination of the spectrum. By employing a vacuum spectroscope and a special photographic plate whose emulsion was very poor in gelatine, he was able to push his observations to wave length 1230. At this point he was stopped by the opacity of the fluorite of which his lenses were made.

I have employed a vacuum spectroscope containing a concave diffraction grating[1] arranged in such a manner that the light path from the source to the photographic plate is wholly in gas. Briefly, the apparatus may be described as a brass tube about 11 cm. in diameter and rather over a meter long. This tube is closed at each end by brass plates ground to fit suitable flanges. In my most recent experiments the light is generated electrically in a discharge tube of quartz provided with tungsten electrodes. This discharge tube fits air tight on one of the two brass plates just mentioned; light from it having passed through a slit, traverses the length of the apparatus and falls upon the diffraction grating by which, having been analyzed into its component colors, it is brought to focus on a special photographic plate placed in close proximity to the slit.

As the discharge tube is in no way separated from the body of the spectroscope it is obviously necessary to choose for the experiment some gas which will not only yield radiations in the region under investigation but which will be transparent to these radiations. My earlier experiments were conducted with hydrogen, since it had shown the necessary characteristics in that part of the spectrum investigated by Schumann. With this gas at a pressure of 2 or 3 mm., and by employing a strong disruptive discharge, about eighteen months ago I was able to extend the spectrum to wave length 900.

A tedious investigation having convinced me that nothing more was to be expected from the use of hydrogen, at least in this neighborhood, I turned my attention to helium, being guided by some of my earlier experiments which had proved that this gas possesses the necessary transparency. At the same time, I made some improvements in my apparatus which, though they left its general form unchanged, resulted in making it far more air tight than ever before. The success of this improvement may be judged by the fact that I have recently been able to leave the spectroscope for over fourteen hours at a pressure of about 3 mm. without being able to detect any leak, either by a McLeod Guage reading to 0.007 mm. or by the appearance of impurities in the spectrum of the gas content. I also took particular pains to purify the helium which I employed.

I have been rewarded for my trouble by a very considerable extension of the spectrum for with helium free from nitrogen, at a pressure of 2 or 3 mm., by the use of the disruptive discharge, and with an exposure of about ten minutes, I have repeatedly observed a number of new lines the most refrangible of which has a wave length of 600. All this has been accomplished with a grating ruled on speculum and with photo-

graphic plates prepared exactly as recommended by Schumann. The wave length 600 represents an extension beyond Schumann's limit greater than that which Schumann himself achieved beyond λ 1850.

The relations of the spectra of hydrogen and helium have recently come into prominence through the theoretical researches of Bohr, Nicholson and others. It so happens that the region on the more refrangible side of wave length 1250 offers an important ground for the study of these relations. In order that the conclusions may be of value, however, it is necessary that the gases under observation should be free from impurities. In the best form of closed discharge tubes the difficulties arising from the contamination of the gas by the material of the tube or by the electrodes, are very considerable, especially when a disruptive discharge is used. In my apparatus, where the brass spectroscope and quartz discharge tube communicate directly, these difficulties are considerably increased. I trust in time to overcome them; but for the present I must confine myself to the following statements:

Prominent in the spectrum of hydrogen is the line at 1216 which forms the first member of a series predicted, on theoretical grounds, by Ritz. I have also found the two next members near wave length 1026 and 972. With hydrogen, they are best seen when a disruptive discharge is used, but with helium containing that trace of hydrogen so difficult to remove, they occur quite strongly with a simple alternating current from a 60 cycle transformer. This is a good illustration of the curious behavior of helium, for an atmosphere of this gas seems sometimes to facilitate the production of the spectra of other gaseous substances which occur in it as impurities. As far as I am able to observe with the uncondensed discharge, all the lines seen with helium are also found in hydrogen. It is only with the disruptive discharge that the new lines on the more refrangible side of λ 900 make their appearance; they occur in helium alone and number seven or eight; some are quite strong.

The wave length of the X-rays as determined by the Braggs, is of the order of one angstrom unit. There is, therefore, a gap of some six hundred units between the region of the ordinary Röntgen rays and the limit which I have reached. Several attempts have been made recently to produce less refrangible X-rays,[2] but there is no satisfactory way of estimating their wave length unless perhaps one relies on the Planck-Einstein formula $Ve = h\nu$. Taking V as the potential drop which gives the cathode particle the velocity necessary to produce a very soft X-ray, e as the charge on the particle and giving to h the usual value ascribed to Planck's constant, the experiments of Dember[3] were made to yield a wave length for the X-ray longer than that which I have

determined. Such speculations, though they are very interesting, cannot be given any great weight. There is still a considerable region between wave length 600 and wave length *one* which must be experimentally explored.

My present limit is set by the adjustment and dimensions of my apparatus. Now that we know that a Schumann plate can be used and that a speculum grating is efficient, I see no insuperable difficulty to a still further extension by purely spectroscopic means.

[1] *The Spectroscopy of the Extreme Ultra-Violet*, Longmans, Greene & Co.

[2] J. J. Thomson, *Phil. Mag., London*, **28**, 620 (1914); E. R. Ladd, *Ann. Physik, Leipzig*, **46**, 605 (1915).

[3] *Berlin, Verh. D. Physik. Ges.*, **15**, 560 (1913).

UNSYMMETRICAL LINES IN TUBE-ARC AND SPARK SPECTRA AS AN EVIDENCE OF A DISPLACING ACTION IN THESE SOURCES

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The most effective laboratory means of causing a displacement of spectrum lines is pressure around the light-source, which has been found to be generally effective in changing the wave-length of lines in the spectra of the arc, spark and electric furnace. These displacements vary in magnitude for different groups of lines at the same pressure. Recently other influences have been found effective in causing small displacements, these manifesting themselves usually in making the spectrum lines unsymmetrical, usually with the red side stronger. A measurement of the maximum of the line under these conditions gives a different wave-length from that obtained for the same line when symmetrical.

We have then the two-fold problem of finding what features in the light-source produce these dissymmetries and the degree to which various lines are subject to such disturbing influences.

About two years ago, the writer described the spectrum of the 'tube-arc,' and noted that many lines were made highly unsymmetrical in this source. The tube-arc is obtained when the tube used in the electric furnace is made thin at its middle so that when the voltage is applied the tube quickly burns apart, forming an arc which carries a very large current at low voltage. This ring of arc is viewed axially and the spectrum shows many peculiarities.

Further use of the tube-arc has now been made in studying the spectrum of iron from λ 3600 to λ 6600, special regard being paid to the dissymmetries produced and the manner in which certain types of lines are intensified.

The tube-arc shows dissymmetries in a very pronounced degree which are present in a more or less incipient state in the ordinary arc, especially in the region near the negative pole. Such dissymmetries are naturally most conspicuous in the case of reversed lines, and but few of these show a fully symmetrical structure in the tube-arc spectrum, the red side usually being stronger than the violet.

The unsymmetrically reversed iron lines fall into two main groups, and these groups show a close connection with the behavior of the lines at various temperatures of the electric furnace and with their pressure displacements. The first group is made up of lines whose dissymmetry in the tube-arc is small or moderate in amount. These appear in the electric furnace at low temperature. The arc under pressure gives them approximately symmetrical and with small or moderate pressure displacements. The lines of the second group are reversed very unsymmetrically in the tube-arc, the red side being often 6 to 8 times as strong as the violet side. They are relatively much stronger in the arc than in the furnace. Under pressure they are very unsymmetrical and show large pressure displacements. In the ordinary iron arc the lines of the second group show a decided dissymmetry near the pole as compared with the center of the arc.

A similar concordance of behavior is shown for a group of strong lines in the green-yellow. These widen toward the violet near the pole of the iron arc and are displaced toward the violet by pressure. In the tube-arc these are unreversed and nearly symmetrical, but show strong displacements toward the violet. Reversal with the violet side strongest has not been observed among the iron lines, but this structure is shown by the tube-arc for the *H* and *K* lines in the calcium spectrum.

The measurement of these tube-arc lines, difficult and uncertain with a micrometer microscope, is made much more accurate by the registering micro-photometer, the curves of which show the structure of the lines and permit the interval between neighboring lines to be measured.

A number of favorably situated lines have been thus measured, the distance being taken from a line approximately symmetrical to one showing strong dissymmetry. The latter line in each case was found to be displaced by amounts ranging from 0.02 to 0.04 A. An interesting feature is that when a line is made narrow by the presence of very

little iron vapor in the tube-arc, the displacement becomes very small, although stronger and wider lines on the same plate retain their one-sided character.

It has long been known that a strongly condensed spark tends to produce unsymmetrical lines in its spectrum. The writer has made a study of this effect using a powerful and very disruptive transformer spark. The dissymmetries produced are strikingly similar to those of the tube-arc. Examination of different regions of the spark showed the dissymmetry to be such as would result from the interior vapor of the spark giving a line displaced toward the red, while the outer vapor gives a symmetrical line coinciding with the usual arc line. Tube-arc lines displaced toward the violet show strong shading on that side in the condensed spark.

For the cause of these dissymmetries, which amount in each case to a displacement of the maximum of the spectrum line, it appears that we must look to an agency other than pressure. Dissymmetries in the spark have sometimes been ascribed to this, but they correspond closely to those of the tube-arc, which is operated in a partial vacuum. Also, in both sources there is no tendency for the displacement to increase in magnitude with increasing wave-length, which is a feature of the pressure effect.

A high vapor density increases the dissymmetry of tube-arc lines, but apparently cannot by itself cause such dissymmetry. Evidence on this point is supplied by the electric furnace, the spectrum lines of which are notably symmetrical with high vapor density. In the arc also, lines may be very wide when much vapor is present, and yet remain symmetrical. When the poles of the arc are approached, however, the occurrence of enhanced lines indicates a condition resembling the spark, and in this region the lines more sensitive to displacement become one-sided, an effect which is magnified in the tube-arc and in the disruptive spark.

At present these effects seem to be harmonized by considering as a necessary condition the presence of electrified particles moving at high velocities, these being given in the arc and spark by the strong potential gradients and in the tube-arc by the enormous consumption of energy by the ruptured carbon tube. If such swiftly moving electrons are present, the effect of their velocities in causing a disturbance of vibration period should be increased by the crowding together due to high vapor density. On this account the relation of the effects here described to the effect of the crowding due to pressure may be a close one, as is borne out by the similarity of the observed displacements.

A combination of electron bombardment with high pressure should give larger displacements for lines which are strengthened by strong electrical conditions. The following evidence bears this out: (*a*) The lines which are strong in the arc relatively to the furnace have large pressure displacements. (*b*) Enhanced lines as a class are found to be displaced by pressure more than arc lines. (*c*) For a given pressure the enhanced lines are displaced more in the spark than in the arc.

The details of this investigation, with corroborative evidence from other spectra, will be published in the *Astrophysical Journal.*

ON THE FACTORIZATION OF VARIOUS TYPES OF EXPRESSIONS

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Not a few mathematicians have dealt with the problem of setting up criteria by means of which the irreducibility of certain expressions in certain domains may be seen at a glance from the character of the expressions. Gauss, Kronecker, Schoenemann, Eisenstein, Dedekind, Floquet, Koenigsberger, Netto, Perron, M. Bauer, and Dumas have written on the subject.[1] The work of these authors may be said to center around the Schoenemann-Eisenstein theorem, which, however, is an exceedingly special case of various theorems obtained (for example, of theorem IV when interpreted for situation 1). With the exception of Floquet and Koenigsberger, the authors mentioned deal exclusively with the case where the expressions are polynomials, and chiefly with the case of polynomials whose coefficients belong to the domain of rational numbers. Floquet and Koenigsberger have extended the investigation to the case of linear homogeneous differential expressions.

A variety of methods have been employed. Thus the theory of algebraic fields, the theory of algebraic functions, the character of the solutions of linear homogeneous differential equations, and even geometric representation (Dumas, loc. cit.) have been used. Elementary methods, requiring no such means, have succeeded in yielding only the less general results.

One of our objects is to show that elementary and comparatively short considerations may be made to yield results more general than any hitherto obtained. In fact, for a complete comprehension of the proofs, little more is required in the way of specific knowledge than an understanding of the definition of the various expressions considered

Moreover, our results have a most intimate connection with the work of every author cited, and, in most cases, the results bearing on the problem before us that are contained in the articles quoted subordinate themselves as special cases to the theorems obtained here.[2]

A sharp distinction has been made in the literature (as regards polynomials) between those investigations that are based on the *divisibility* properties of the coefficients and those that proceed from the consideration of the *magnitude* of the coefficients.[3] One of the interesting results is that the gap between these two types of investigation may, in a certain sense, be bridged. As a consequence, it is possible to show how the various theorems obtained flow from certain general considerations as a common source, and thus a surprising unification of the material is achieved.[4]

Our work deals with polynomials (for the various kinds of coefficients considered, see situations 1–5 incl.), linear homogeneous differential expressions, linear homogeneous difference expressions (for the first time as regards our problem, as far as the writer knows), and more general expressions. There is no better way of making what is essential in the proofs come to the front than by treating the subject in abstract, postulational fashion. Thereby also the interconnection between the results and the unification above referred to are made manifest. Thus, it is easy to see, as Koenigsberger has pointed out by an example, that the Schoenemann-Eisenstein theorem cannot be directly extended to the case of linear homogeneous differential expressions; but our abstract treatment furthermore lays bare the underlying reason—by no means evident otherwise—why it breaks down, and at once indicates what analogous theorem may replace it.[5] The abstract treatment is, moreover, especially fitting here because a small number of simple assumptions is sufficient for the foundations of the theory.

We start[6] with any *aggregate* $\mathfrak{S}$ whatsoever and shall deal with finite, ordered subaggregates $E = (e_0, e_1, e_2, \cdots, e_m)$ of $\mathfrak{S}$, $e_\nu \{\nu = 0, 1, 2, \cdots, m\}$ being an element of $\mathfrak{S}$.

Such a finite, ordered subaggregate E we shall call a '*parenthesis*' of $\mathfrak{S}$; m will be called the '*order*' of E. We assume that the '*product*' $A \cdot B = (a_0, a_1, a_2, \cdots, a_r) \cdot (b_0, b_1, b_2, \cdots, b_s)$ of any two parentheses of $\mathfrak{S}$ is equal to a parenthesis $C = (c_0, c_1, c_2, \cdots, c_n)$ of $\mathfrak{S}$ for which $n = r + s$. We assume furthermore that with every element e of $\mathfrak{S}$ there is associated a single number η, called the '*rank* of e,' η being an integer or $-\infty$ (never $+\infty$), such that when $A \cdot B = C$ one of the following 3 sets of relations holds (see situations 1–10 that make either I or II or III valid):

$$\text{I}\quad a)\quad \gamma_\nu \leqq \max_{\lambda+\mu=\nu}(\alpha_\lambda+\beta_\mu),\quad \{\nu=0,1,2,\cdots,n\},$$

$$b)\quad \gamma_\nu = \max_{\lambda+\mu=\nu}(\alpha_\lambda+\beta_\mu),\quad \{\nu=0,1,2,\cdots,n\},$$

if $\alpha_\lambda+\beta_\mu$ attains its maximum, λ and μ varying so as to satisfy the relation $\lambda+\mu=\nu$, for a single pair of values of λ and μ. Here $\alpha_\lambda, \beta_\mu, \gamma_\nu =$ rank[7] of a_λ, b_μ, c_ν.

$$\text{II}\quad a)\quad \gamma_\nu \leqq \max_{\lambda+\mu\leqq\nu}(\alpha_\lambda+\beta_\mu),\quad \{\nu=0,1,2,\cdots,n\},$$

$$b)\quad \gamma_\nu = \max_{\lambda+\mu=\nu}(\alpha_\lambda+\beta_\mu),\quad \{\nu=0,1,2,\cdots,n\},$$

if $\alpha_\lambda+\beta_\mu$ attains its maximum M_ν (λ and μ varying so as to satisfy $\lambda+\mu=\nu$) for a single pair of values of λ and μ, and if $M_\nu \geqq \alpha_\lambda+\beta_\mu$ whenever $\lambda+\mu<\nu$.

$$\text{III}\quad a)\quad \gamma_\nu \leqq \max_{\lambda+\mu+\sigma=\nu}(\alpha_\lambda+\beta_\mu+\sigma),\quad \{\nu=0,1,2,\cdots,n\},$$

σ being an integer such that $0\leqq\sigma\leqq\nu$.

$$b)\quad \lambda_\nu = \max_{\lambda+\mu=\nu}(\alpha_\lambda+\beta_\mu),\quad \{\nu=0,1,2,\cdots,n\},$$

if $\alpha_\lambda+\beta_\mu$ attains its maximum M_ν (λ and μ varying so as to satisfy $\lambda+\mu=\nu$) for a single pair of values of λ and μ, and if $M_\nu > \alpha_\lambda+\beta_\mu+\sigma$ for $\lambda+\mu+\sigma=\nu$ and $\sigma>0$.

It is to be noted that I implies II and that II implies III. *Hence every theorem proved for all* $\mathfrak{S}^{\text{III}}$ (*i.e., for all* $\mathfrak{S}$ *having property* III) *is valid for every* $\mathfrak{S}^{\text{II}}$ *and* $\mathfrak{S}^{\text{I}}$; *and every theorem proved for all* $\mathfrak{S}^{\text{II}}$ *holds for every* $\mathfrak{S}^{\text{I}}$.

We shall now describe various important situations where I or II or III holds. For this purpose, we must define in each case the *aggregate* $\mathfrak{S}$, the *parentheses* of $\mathfrak{S}$, the *product* of two parentheses and the *rank* of every element of $\mathfrak{S}$.

I *holds in the following situations* (1–7 *incl.*):

1. $\mathfrak{S}$ consists of the set of rational numbers. We understand by the parenthesis $(e_0, e_1, e_2, \cdots, e_m)$ of $\mathfrak{S}$ the rational polynomial $e_0y^m+e_1y^{m-1}+\cdots+e_m$ in the letter y. The product of two parentheses $(a_0, a_1, \cdots, a_r)\cdot(b_0, b_1, \cdots, b_s)$ is defined, as usual, to be equal to the parenthesis $(c_0, c_1, \cdots, c_n)$, where $n=r+s$ and $c_0=a_0b_0$, $c_1=a_0b_1+a_1b_0$, $\cdots$, $c_n=a_rb_s$. The rank of an element $e=e'/e''$ (where e' and e'' are integers) is defined with reference to a fixed prime p. First let $e\neq 0$; let e' be divisible by $p^{d'}$ but not by $p^{d'+1}$; e'', by $p^{d''}$ but not by $p^{d''+1}$. We define the rank of e by the equation $\eta=d''-d'$. Moreover, we (naturally) define the rank of 0 to be $-\infty$.

2. $\mathfrak{S}$ consists of the class of the Hensel p-adic numbers. The parentheses of $\mathfrak{S}$ and the product of two parentheses are defined as in 1. The rank of the p-adic number e is the negative of Hensel's[8] 'order' of e with respect to p.

3. $\mathfrak{S}$ consists of the class of rational fractions $e=e'(x_1, \cdots, x_K)/e''(x_1, \cdots, x_k)$ in k letters $x_1, x_2, \cdots, x_k$, e' and e'' being polynomials with

arbitrary complex numerical coefficients.[9] The parentheses and the product of two parentheses are defined as in 1. The rank η of an element e ($\neq 0$) is defined as $d'-d''$, where d' and d'' represent respectively the degrees (in the usual sense) of e' and e'' in the k letters $x_1, \cdots, x_k$. The rank of 0 is defined to be $-\infty$.

4. $\mathfrak{S}$ consists of the collectivity of elements $e = \sum_{\lambda=0}^{\infty} c_\lambda x^{\eta-\lambda}$, where the c's are arbitrary complex numbers, that is, e is a Laurent series having only a finite number of terms with positive exponents.[10] The parentheses and the products of parentheses are defined as in 1. The rank of e ($\neq 0$) is defined to be the exponent of x in the first non-zero term of the development of e. The rank of 0 is (naturally) defined to be $-\infty$.

5. $\mathfrak{S}$ consists of the collectivity of fractions $e = e'/e''$, where e' and e'' are power series in k letters $x_1, x_2, \cdots, x_k$ with arbitrary complex coefficients:

$$e' = \sum_{\lambda_1, \lambda_2, \cdots, \lambda_k = 0}^{\infty} c'_{\lambda_1, \lambda_2, \cdots, \lambda_k} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}, \quad e'' = \sum_{\lambda_1, \lambda_2, \cdots, \lambda_k = 0}^{\infty} c''_{\lambda_1, \lambda_2, \cdots, \lambda_k} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}.$$

The rank of e ($\neq 0$) is defined as $d''-d'$, where d' and d'' represent the lowest degrees (in the usual sense) in the k letters $x_1, x_2, \cdots, x_k$ of a non-zero term of e' and e'' respectively. The rank of 0 is $-\infty$.

6. $\mathfrak{S}$ consists of all rational fractions $e(x) = e'(x)/e''(x)$ in x ($e'(x)$ and $e''(x)$ being polynomials) with arbitrary complex coefficients. The parenthesis $(e_0, e_1, \cdots, e_m)$ of $\mathfrak{S}$ is the linear homogeneous difference expression $e_0 y_{x+m} + e_1 y_{x+m-1} + \cdots + e_m y_x$. The product $A \cdot B = (a_0, a_1, \cdots, a_r) \cdot (b_0, b_1, \cdots, b_s)$ of two parentheses of $\mathfrak{S}$ is the ordinary *symbolic* product of the linear difference expressions A and B and is equal to $(c_0, c_1, \cdots, c_n)$ where $n = r + s$ and

$$c_\nu = \sum_{\lambda=0}^{\nu} a_\lambda(x)\, b_{\nu-\lambda}(x+r-\lambda), \qquad \{\nu = 0, 1, 2, \cdots, n\}.$$

The rank of $e(x)$ is what is ordinarily called the degree of $e(x)$, i.e., $d'-d''$, where d' = degree of $e'(x)$ and d'' = degree of $e''(x)$.

7. $\mathfrak{S}$ and rank are defined as in 4; parentheses and products of parentheses, as in 6.

II *holds in the following situations* (8–9 *incl.*):

8. $\mathfrak{S}$ and rank are defined as in 6. The parenthesis $(e_0, e_1, \cdots, e_m)$ of $\mathfrak{S}$ is the linear homogeneous differential expression

$$e_0(x)\frac{d^m y}{dx^m} + e_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + e_m(x)\, y.$$

The product $A \cdot B = (a_0, a_1, \cdots, a_r) \cdot (b_0, b_1, \cdots, b_s)$ of two parentheses is the ordinary *symbolic* product of the linear differential expressions A and B and is equal to $(c_0, c_1, \cdots, c_n)$, where $n = r+s$ and

$$c_\nu = \sum_{\mu=0}^{\nu} \sum_{\lambda=0}^{\nu-\mu} \binom{r-\lambda}{\nu-\mu-\lambda} a_\lambda \frac{d^{\nu-\mu-\lambda} b_\mu}{dx^{\nu-\mu-\lambda}},$$

$$\left\{ \binom{r-\lambda}{\nu-\mu-\lambda} = \frac{(r-\lambda)!}{(\nu-\mu-\lambda)!\,(r-\nu+\mu)!} \right\}.$$

9. $\mathfrak{S}$ and rank are defined as in 4; parentheses and products of parentheses, as in 8.

III *holds in the following situation*:

10. $\mathfrak{S}$ consists of the collectivity of fractions $e = e'/e''$, where e' and e'' are power series in $x-h$ $(=x_1$, cf. 5) with arbitrary complex coefficients. The rank of e is defined as in 5. The parentheses of $\mathfrak{S}$ and products of parentheses are defined as in 8.

We shall now proceed to the statement of certain consequences of the assumptions made that relate to factorization properties of a given parenthesis $C = (c_0, c_1, \cdots, c_n)$ of $\mathfrak{S}$. We base all further considerations on the tentative assumption that C may be expressed as a product $A \cdot B = (a_0, a_1, \cdots, a_r)\ (b_0, b_1, \cdots, b_s)$ of two parentheses, where $r \geqq 1$ and $s \geqq 1$. When other assumptions to be made in the theorems contradict this assumption of 'reducibility,' it must be that C is under those later assumptions incapable of being expressed as a product of two parentheses whose orders are at least 1. C is then said to be 'irreducible.' (In general, the terminology employed for our abstract situation is parallel to that for the ordinary concrete situations.) *We assume furthermore throughout in what follows that* γ_0, *the rank of* c_0, *is finite* (i.e., $\neq -\infty$). We introduce the following notations (partly for the purpose of indicating our method of investigation and partly for the purpose of simplifying certain future statements):

$$\begin{cases} \Delta_0 = \alpha_0 - \alpha_0 = 0, & \Delta_1 = \alpha_1 - \alpha_0, \cdots, & \Delta_r = \alpha_r - \alpha_0. \\ \Delta_0' = \beta_0 - \beta_0 = 0, & \Delta_1' = \beta_1 - \beta_0, \cdots, & \Delta_s' = \beta_s - \beta_0. \end{cases}$$

$$\begin{cases} G = \underset{1 \leqq \nu \leqq r}{max} (\Delta_\nu);\ \Delta_\tau = \text{last } \Delta_\nu\ \{\nu = 1, 2, \cdots, r\} \text{ equal to } G. \\ G' = \underset{1 \leqq \nu \leqq s}{max} (\Delta_\nu');\ \Delta_{\tau'}' = \text{last } \Delta_\nu'\ \{\nu = 1, 2, \cdots, s\} \text{ equal to } G'. \end{cases}$$

$$\begin{cases} M = \underset{1 \leqq \nu \leqq r}{max} \left(\frac{\Delta_\nu}{\nu}\right);\ \frac{\Delta_i}{i} = \text{last } \frac{\Delta_\nu}{\nu}\ \{\nu = 1, 2, \cdots, r\} \text{ equal to } M. \\ M' = \underset{1 \leqq \nu \leqq s}{max} \left(\frac{\Delta_\nu'}{\nu}\right);\ \frac{\Delta_{i'}'}{i'} = \text{last } \frac{\Delta_\nu'}{\nu}\ \{\nu = 1, 2, \cdots, s\} \text{ equal to } M'. \end{cases}$$

The following results (lemmas[11] I–VI *incl., theorems* I–X *incl.) are valid for every* $\mathfrak{S}^{II}$:

Lemma I. *If* $G \leqq 0$ *and* $G' \leqq 0$, $\gamma_\nu - \gamma_0 \leqq 0$ *for every* ν.

Lemma II. *If* $G \geqq 0$ *and* $G' \geqq 0$, $\gamma_{\tau+\tau'} - \gamma_0 = G + G' \geqq \gamma_\nu - \gamma_0$ *for every* ν.

Lemma III. *If* $M > M'$ *and* $M > 0$ $(M' > M$ *and* $M' > 0)$, $M = \frac{\gamma_t - \gamma_0}{t} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ $\left(M' = \frac{\gamma_{t'} - \gamma_0}{t'} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}\right)$ *for every* $\nu \neq 0$, *the* $>$ *sign alone holding when* $\nu > t$ $(\nu > t')$.

Lemma IV. *If* $M > M' \geqq 0$ $(M' > M \geqq 0)$, *there is no fixed* k *such that* $\gamma_k > \gamma_\nu$ *and* $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0, k$.

Lemma V. *If* $M > M' > 0$ $(M' > M > 0)$ *there is no fixed* k *such that* $\gamma_k \geqq \gamma_\nu$ *and* $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu > 0$.

Lemma VI. *If* $M = M' > 0$, $\frac{\gamma_{t+t'} - \gamma_0}{t+t'} = M = M' \left(= \frac{\Delta_t}{t} = \frac{\Delta'_{t'}}{t'}\right) \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu > 0$, *the* $>$ *sign alone holding when* $\nu > t + t'$.

Theorem I. *If, for a fixed* k, $\gamma_k - \gamma_0 > 0$ and $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n\}$, *at least one of the following* $n - k + 1$ *congruences holds:*

$$\gamma_k - \gamma_0 \equiv 0 \left(\mathrm{mod}\, \frac{k}{(k, \sigma)}\right),$$

where σ *takes the successive values* $k - r$, $k - r + 1$, $\cdots$, $n - r$ *(or* $k - s$, $k - s + 1$, $\cdots$, $n - s$*) and* (k, σ) *represents, as usual, the G.C.D. of* k *and* σ.

Theorem II. *If, for a fixed* k, $\gamma_k - \gamma_0 > 0$, $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n\}$ *and* $(\gamma_k - \gamma_0, k) = 1$, *the parenthesis* C *contains an irreducible factor of order* $\geqq k$.

Theorem III (a special case of I). *If* $\gamma_n - \gamma_0 > 0$ *and* $\frac{\gamma_n - \gamma_0}{n} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n-1\}$, $\gamma_n - \gamma_0 \equiv 0 \left(\mathrm{mod}\, \frac{n}{(n, r)}\right)$.

Theorem IV (a special case of III). *If* $\gamma_n - \gamma_0 > 0$, $\frac{\gamma_n - \gamma_0}{n} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ $\{\nu = 1, 2, \cdots, n-1\}$ *and* $(\gamma_n - \gamma_0, n) = 1$, C *is irreducible.*

Theorems V–X inclusive materially extend and generalize Perron's theorem, *J. Math., Berlin*, **132**, 304 (1907), which Perron designates as 'ein sehr allgemeines Kriterium.'

Theorem V. *If* $G \geqq 0$, $G' \geqq 0$, $\gamma_k > \gamma_\nu$ *for every* $\nu \neq k$ *and* $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *the following relations hold:*

$$\begin{cases} 0 < M = M' = \dfrac{\gamma_k - \gamma_0}{k} \left(= \dfrac{\Delta_t}{t} = \dfrac{\Delta'_{t'}}{t'} \right); \\ k = t + t';\ t = \tau,\ t' = \tau';\ (\Delta_t = G;\ \Delta_{t'} = G'). \end{cases}$$

Theorem VI. *If* $G \geqq 0$, $G' \geqq 0$, $\gamma_k > \gamma_\nu$ *for every* $\nu \neq k$, $\frac{\gamma_k - \gamma_0}{k} > \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0, k$, *and* $(\gamma_k - \gamma_0, k) = 1$, C *is irreducible.*

Theorem VII. *If* $G > 0$, $G' > 0$, $\gamma_k \geqq \gamma_\nu$ *for every* ν *and* $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *the same conclusions may be drawn as in theorem* V *with the possible exception of* $t = \tau$ *and* $t' = \tau'$.

Theorem VIII. *If* $G > 0$, $G' > 0$, $\gamma_k \geqq \gamma_\nu$ *for every* ν, $\frac{\gamma_k - \gamma_0}{k} > \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0, k$, *and* $(\gamma_k - \gamma_0, k) = 1$, C *is irreducible.*

Theorem IX (generalizes VI). *If* $G \geqq 0$, $G' \geqq 0$, $\gamma_k > \gamma_\nu$ *for every* $\nu \neq k$, $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *and if furthermore every parenthesis* $(e_0, e_1, \cdots, e_m)$ *that may occur as a factor of* C *is such that* $\eta_\nu \geqq \eta_0$ *for at least one* $\nu > 0$, *every decomposition of* C *into a product of parentheses contains at most* $(\gamma_k - \gamma_0, k)$ *factors.*

Theorem X (generalizes VIII). *If* $G > 0$, $G' > 0$, $\gamma_k \geqq \gamma_\nu$ *for every* ν, $\frac{\gamma_k - \gamma_0}{k} \geqq \frac{\gamma_\nu - \gamma_0}{\nu}$ *for every* $\nu \neq 0$, *and if furthermore every parenthesis* $(e_0, e_1, \cdots, e_m)$ *that may occur as a factor of* C *is such that* $\eta_\nu > \eta_0$ *for at least one* ν, *every decomposition of* C *into a product of parentheses contains at most* $(\gamma_k - \gamma_0, k)$ *factors.*

The lemmas leading up to theorems XI and XII are omitted.

Theorem XI. *Theorem* I *holds for every* $\mathfrak{S}^{\mathrm{I}}$ *if the inequality* $\gamma_k - \gamma_0 > 0$ *is replaced by* $\gamma_k - \gamma_0 < 0$.

Theorem XII. *Theorem* I *holds for every* $\mathfrak{S}^{\mathrm{III}}$ *if the inequality* $\gamma_k - \gamma_0 > 0$ *is replaced by* $\frac{\gamma_k - \gamma_0}{k} > 1$.

The results here outlined will be offered for publication *in extenso* to the *Transactions of the American Mathematical Society.*

[1] Gauss, *Disquisitiones arithmeticae* (1801) Art. 341; Kronecker, *J. Math., Berlin*, 29, 280 (1845), 100, 79 (1887), *J. Math., Paris*, 19, 177 (1854) and ser. 2, 1, 399 (1856); Schoenemann, *J. Math., Berlin*, 32, 100 (1846) and 40, 188 (1850); Eisenstein, *J. Math., Berlin*, 39, 160 (1850); Dedekind, *J. Math., Berlin*, 54, 27 (1857); Floquet, *Ann. Sci. Ec. norm., Paris*, (1879), cited by Koenigsberger; Koenigsberger, *J. Math., Berlin*, 115, 53 (1895), 121, 320 (1900), *Math. Ann., Leipzig*, 53, 49 (1900); Netto, *Math. Ann.*, 48, 81 (1897); Perron, *Math. Ann.*, 60, 448 (1905), *J. Math., Berlin*, 132, 288 (1907); M. Bauer, *J. Math., Berlin*, 128, 87 and 298, (1905), 132, 21 (1907), 134, 15 (1908); Dumas, *J. Math., Paris*, ser. 6, 2, 191 (1906).

[2] Thus the work on the subject before the publication of the Schoenemann-Eisenstein theorem is summarized and generalized by that theorem, which, as previously indicated, is a very special case of theorem IV. In fact, theorem I alone, for example, includes as special cases a great bulk of the results heretofore published, and, in particular, the following: the theorem of Floquet—quoted by Koenigsberger—, all the results contained in the 1895 paper of Koenigsberger, nearly all contained in his 1900 *Math. Ann.* paper, almost all the results of Netto and almost all the results in the 1905 paper of Perron. Cf. also the paragraph preceding the statement of theorem V.

[3] Perron, *J. Math., Berlin*, 132, 288, and Loewy, in Pascal's *Repertorium* (1910) Analysis I_1, 292 and 293.

[4] I may perhaps be permitted to remark, for the purpose of indicating that the results given are less artificial than one would at first suppose, that I had obtained my chief results with little knowledge of the literature. Their intimate connection with results already obtained points to a degree of 'naturalness' of these results that one would hardly attribute to them in the absence of such a connection.

[5] As a matter of fact, two distinct theorems obtained may be properly regarded as (highly) generalized Schoenemann-Eisenstein theorems for the case of linear homogeneous differential expressions: theorem I for situation 8 and theorem XII for situation 10. Curiously, Koenigsberger himself has obtained theorems—special cases of XII—for differential expressions that may be properly regarded as generalized Schoenemann-Eisenstein theorems, without his having noticed this relation.

[6] It is possible to build up just as general a theory as ours by dealing exclusively with 'parentheses' whose elements are 'ranks'—see below in the same paragraph—and hence always integers or $-\infty$. The reader who prefers a more concrete, tho necessarily less general, discussion may, for example, at once interpret $\mathfrak{S}$, '*parenthesis*,' '*product*' and '*rank*'—see below in the same paragraph—as in situation 8. In that case, II will hold.

[7] In general, we denote the rank of an element represented by a Latin letter by the corresponding Greek letter. By $\max_{\lambda+\mu=\nu}(\alpha_\lambda+\beta_\mu)$, $\max_{\lambda+\mu\leqq\nu}(\alpha_\lambda+\beta_\mu)$, $\max_{\lambda+\mu+\sigma=\nu}(\alpha_\lambda+\beta_\mu+\sigma)$—see II and III below—we naturally understand the largest value attained by the numbers of the set $\alpha_\lambda+\beta_\mu$, $\alpha_\lambda+\beta_\mu$, $\alpha_\lambda+\beta_\mu+\sigma$, λ, μ and σ varying so as to satisfy the relations $\lambda+\mu=\nu$, $\lambda+\mu\leqq\nu$, $\lambda+\mu+\sigma=\nu$, and in addition, of course, $0\leqq\lambda\leqq r$ and $0\leqq\mu\leqq s$.

[8] Hensel, *J. Math., Berlin*, 127, 51–84, §2 (1904) or Zahlentheorie (1913), chaps. 3 and 6.

[9] More generally, the numerical coefficients may belong to any abstract system $(K,+,\times)$, where K is a class, such that if a and b are elements of K both $a+b$ and $a\times b$ are elements of K. This remark applies just as well to the numerical coefficients in situations 4–10 incl.

[10] The question of convergence does not enter here because the formal character of the series is sufficient for our purpose. More generally, we may have such series in two or more variables, the required change in the definition of rank being evident.

[11] These lemmas lead up to the theorems I–X and are given partly for the purpose of indicating the nature of the proofs and partly because they are believed to be of interest in themselves.

THE DIRECTION OF ROTATION OF SUN-SPOT VORTICES

By George E. Hale

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Read before the Academy, December 8, 1914. Received, April 15, 1915

Whirling storms in the earth's atmosphere, from the extensive and moderate cyclone to the small and destructive tornado, follow a well-known law of rotation: right-handed or clockwise in the southern hemisphere and counter-clockwise in the northern. These directions are easily explained by the increasing eastward velocity of the air from pole to equator. As sun-spots are vortex phenomena, analogous in many respects to tornadoes, it is interesting to inquire as to the law of their rotation, since this may throw light on their nature and origin.

It is a well-known fact that when a normal Zeeman triplet is observed along the lines of force of a magnetic field, the central component is absent, and the two side components are circularly polarized in opposite directions. A quarter-wave plate and Nicol prism mounted over the slit of the spectroscope permit either component to be cut off at will. Furthermore, if the polarizing apparatus be adjusted so as to extinguish one component, reversal of the current through the coils of the magnet will cause this component to reappear, while the other will be extinguished. The method thus offers a simple means of determining the polarity of a magnetic field, which can still be applied when the angle between the line of sight and the lines of force is as great as 60° or 70°. In this case, however, the central line of the triplet is present, and the elliptically polarized light of the side components can be only partially extinguished.

Every sun-spot exhibits a magnetic field, whose polarity is determined by the direction of rotation of the electrons in the spot vortex. To learn the polarity, and hence the direction of rotation, it is therefore only necessary to observe whether the red or the violet component of a spot triplet is transmitted by the polarizing apparatus.

In my earlier work attention was concentrated on a few of the largest spots, in order to secure unquestionable evidence as to the existence of the Zeeman effect. The possibility of finding a law of rotation appeared to be remote, as spots of opposite polarity were observed in the same hemisphere. Subsequently, when the characteristics of bipolar spot groups had been discovered, the search for a rotation law was renewed, with the results given in this paper.

The typical sun-spot group consists of two spots of opposite magnetic polarity, lying on a line which usually makes only a small angle with the

solar equator. One of the spots may be replaced by several smaller ones, or even be absent altogether, its place in the latter case being indicated by calcium and hydrogen flocculi. Small companion spots, of either polarity, may attend either member of the pair. The characteristic feature of the group is the presence of magnetic fields, of opposite polarity, in the regions including its eastern and western extremities.

Deferring until another occasion a discussion of the probable origin of bipolar groups, it will suffice for present purposes to emphasize the necessity of recognizing the existence and importance of such pairs. The following quotation from a valuable paper by Father Cortie, based upon a study of some 3500 drawings of spots made at Stonyhurst College Observatory, is of special significance in this connection:

> The chief type, however, of which the above mentioned are in most, possibly in all, cases but phases, is the double spot formation, with a train of smaller spots between the two principal spots of the group, the whole group generally drifting into more or less parallelism with the solar equator. In this form the principal spot, which eventually becomes a normal spot of regular outline, is generally the leading spot, but in many cases it is the following spot, while sometimes the preponderance in area alternates between the two, as the group traverses the disk. In yet rarer instances both the chief spots develop as regular spots. (*Astrophys. J.*, **13**. 261, 1901.)

As the preceding (western) spot of a bipolar group usually lasts longer than the following spot, and is likely to be the larger of the two, we may take it as the dominant member of the group, and classify the polarities of northern and southern spots accordingly. Single spots, followed by a train of flocculi, are classified in Table 1 as preceding spots. Under the headings V and R are given the number of spots for which the "marked strip" of the compound quarter-wave plate transmitted the violet or red component respectively. The spots of the old and new cycles (observed before and after the recent sun-spot minimum) are tabulated separately.

TABLE 1

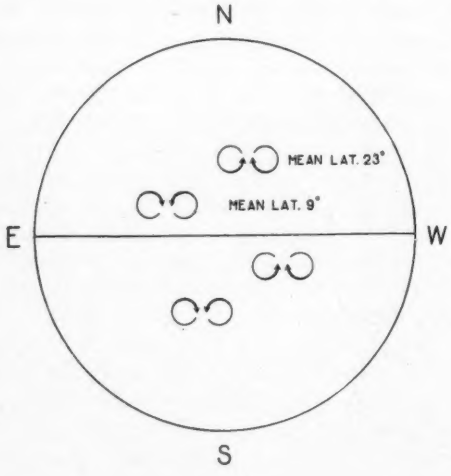
HEMISPHERE	OLD CYCLE				NEW CYCLE			
	Pr. Spots		Foll. Spots		Pr. Spots		Foll. Spots	
	V	R	V	R	V	R	V	R
N	5	0	0	1	0	13	10	0
S	2	16	8	0	9	1	1	7

TABLE 2

	OLD CYCLE		NEW CYCLE	
	N	S	N	S
Mean ϕ and No. of spots	10° (7)	9° (22)	22° (11)	26° (6)
Weighted Mean	9.0° (29)		23.2° (17)	

From this summary it appears that the five preceding spots observed in the northern hemisphere before the minimum were characterized by the violet component, while the single following spot gave the red component. In the southern hemisphere, with two exceptions, the polarities were of the opposite sign.

After the minimum, to my surprise, the polarities were found to be reversed in both hemispheres, as the tabulated results for the new cycle indicate. The explanation of this fact probably lies in the change of latitude which a new cycle introduces. While the last spots of the old cycle appear in low latitudes, the first spots of the new cycle occur



much farther from the equator. Thus we apparently have on the sun two spot zones, in which the great majority of spot vortices rotate in opposite directions. The approximate mean latitudes of these zones are shown in Table 2.

From the best supplementary evidence at present available, the true direction of rotation of a preceding spot vortex in the low latitude zone is counter-clockwise, corresponding to that of a terrestrial tornado. The opposite direction obtains in the southern hemisphere, as on the earth. But in high latitudes the signs are reversed—a result which is likely to prove significant in future studies of the sun (see figure). The complete details of this investigation will be published in the *Astrophysical Journal*.

SOME VORTEX EXPERIMENTS BEARING ON THE NATURE OF SUN-SPOTS AND FLOCCULI

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Read before the Academy, April 20, 1915. Received, May 22, 1915

The following working hypothesis of sun-spots was first proposed, in a somewhat different form, in the Annual Report of the Director of the Mount Wilson Solar Observatory for 1912:

As the result of an eruption, or some other cause tending to produce rapid convection, a gaseous column moves upward from within the sun toward the surface of the photosphere. Vortex motion is initiated by differences in velocity of adjoining surfaces or by irregularities of structure and is maintained by convection. The circulation in the vortex is vertically upward and then outward along the photosphere, as in a terrestrial tornado. Expansion produces cooling at the center of the vortex, and a comparatively dark area (the umbra) results. As in Harker's electric furnace experiments, a rapid flow of negative ions sets in toward the cooler gases at the center from the hotter gases without. These ions, whirled in the vortex, produce a magnetic field, plainly indicated by the widened and resolved lines of the sun-spot spectrum.

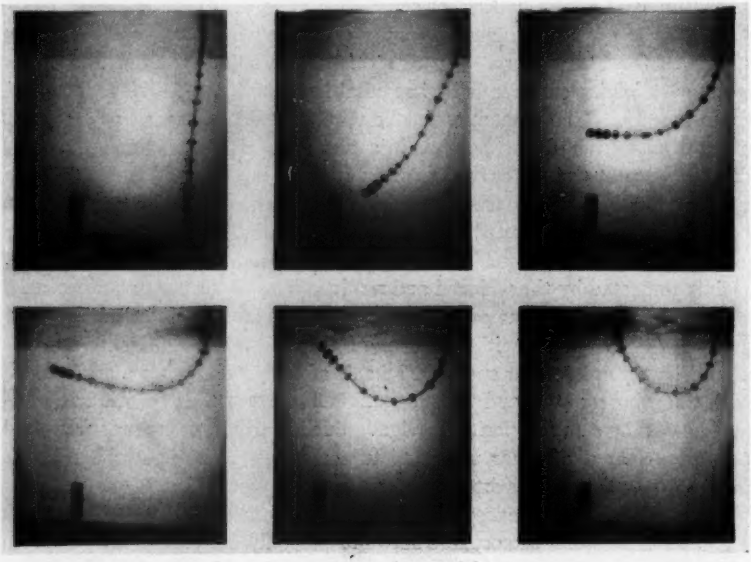
Hitherto we have assumed the spot to be single. The typical sun-spot group is double, though either or both of the principal members may be replaced by several smaller spots. The axis of the group makes a small angle with the solar equator, and the magnetic polarities of the preceding and following members are opposite in sign. Such a group may result from the tendency of any columnar vortex to turn up toward the photosphere, thus forming a semi-circular vortex ring.

A secondary vortex, caused by the influence of the low-lying spot vortex on the gases of the solar atmosphere above it, partially determines the structure of the hydrogen flocculi, which are drawn inward at high levels, then downward and outward at the upper levels of the spot vortex. The hydrodynamic structure thus developed may be more or less modified by the influence of the spot magnetic field on the trajectories of the moving ions within it.

The presence of magnetic fields in all sun-spots and the spectroscopic observations of Evershed and St. John on the revolution of the spot vapors at various levels, seem to prove beyond doubt that sun-spots are vortices. The object of the present paper is to describe some recent experiments bearing on the suggested explanation of double or multiple spots, and the flocculi associated with them.

A closely wound helix of brass wire, with disks of wood threaded on it to increase friction, is hung vertically from the shaft of an electric motor, and spun at moderate velocity in water. A columnar vortex is

thus set up with the helix as a core, which will remain nearly vertical as long as its angular velocity is low. But if the speed of the motor is increased, the lower end of the helix will gradually move toward the surface, giving rise to the successive forms shown[1] in figures 1 to 6. The semi-circular vortex ring finally produced, like an ordinary smoke ring, will have a motion of translation at right angles to its plane, in a direction corresponding to the direction of rotation of its inner edge. It is interesting to observe the depressions and whirls in the water when the end of the helix is approaching the surface. These represent in the hy-



pothesis the incipient spots which often appear and disappear near one extremity of a bipolar group.

An instructive control of this experiment may be readily applied. The water in the tank is set into rotation (by an auxiliary paddle) about the axis of the helix before the motor is started. After the helix is set spinning it will turn up much more rapidly than in quiet water if the direction of the preliminary rotation is opposed to that of spin, i.e., if the water is whirling in the same direction as the second 'spot' of the group. If, however, the whole body of water is rotating in the same direction as the helix, the latter will turn up with difficulty, or remain in the vertical position without turning up at all.

Thus a sun-spot vortex, if produced within a region already rotating

in the same direction, would tend to retain a columnar form, and the second spot of the group would not appear. Other conditions favorable to the production of single spots are low angular velocity of the vortex (similar in effect to the above case), a large ratio of diameter to length, and the presence within the sun of persistent eruptive centers, such as are suggested by the frequent recurrence of spots in the same region of the photosphere. It should be noted that convection may be a factor in prolonging the life of single spots.

If bipolar spots are semi-circular vortices, they should exhibit the proper motion characteristic of vortex rings. This motion, as already remarked, is at right angles to the plane of the ring, and in a direction corresponding with that of its inner edge. In a paper read before the Academy at its last autumn meeting,[2] it was shown that high and low latitude bipolar spot groups rotate in opposite directions. These directions are such that low latitude groups should move toward the equator, while high latitude groups should move toward the poles.

An investigation of the motion of spots in latitude was made many years ago by Carrington, who says:

> In the above table it will be remarked that there is more distinctly a trace of motion in latitude, the signs being on the whole + for latitudes higher N. or S. than 20°, though the daily polar motion between 20° and 40° of latitude on an average does not exceed 2′, a quantity which could only be deduced from the totality of a large number of single results. Between the parallels of 10° to 20° the motion in latitude is evidently very small; but the signs are generally negative and a feeble tendency towards the equator of about 1′ per diem is indicated. Within 10° of the Equator on either side no reliable motion in latitude appears to exist, the signs varying much and the mean results being of less weight.[3]

These results relate to all spots, both single and multiple, whose motions were observed by Carrington. His conclusion as to the reversed directions of motion of high and low latitude spots has recently been confirmed by Dyson and Maunder, who discussed the extensive observational material comprised in the Greenwich sun-spot measures from 1874 to 1912. They remark:

"A slight general tendency is suggested for high-latitude spots in either hemisphere to move away from the equator, but for those in lower latitudes to move toward it."[4]

This apparent confirmation of the hypothesis should not be taken too seriously, as it remains to be seen whether a closer comparison, excluding single spots and grouping the results in strict conformity with the observed polarities of bipolar pairs, would be equally favorable.

The need of caution is emphasized by the fact that the observed daily motions in latitude appear to be considerably smaller than theory would indicate. We have verified Kelvin's approximate expression for the motion of translation of vortex rings[5] by means of a semi-circular vortex ring in water, with paddles which can be driven at angular velocities varying through a wide range. But it does not follow that the same expression can be applied under the very different conditions existing in the sun, and we are seeking further information regarding this point before attempting to complete the quantitative test.

We may now inquire whether experimental methods will not throw some light on the difficult problem of the structure of the hydrogen flocculi. In 1908 we found that sun-spots are centers of vortex phenomena, plainly shown by the spectroheliograph when the sun is photographed with the light of the red line ($H\alpha$) of hydrogen. It was subsequenly proved by Evershed and St. John, however, that these are secondary vortices, extending down through the solar atmosphere from the hydrogen level, where their stream-lines are visible, toward an underlying vortex which constitutes the spot and produces its magnetic field. Are these secondary vortices of hydrodynamic origin, as St. John has maintained, or are they due to the effect of the magnetic field on the trajectories of the electrically charged particles in the solar atmosphere? The latter view, implied in Brester's aurora hypothesis of the flocculi and more clearly stated by Deslandres, has been fully developed mathematically by Störmer. Undoubtedly both hydrodynamic and electromagnetic phenomena must influence the structure of the solar atmosphere, but a satisfactory means of distinguishing between the two has been lacking.

The structure of the hydrogen flocculi shown on spectroheliograph plates can be more or less closely imitated with the aid of smoke in a closed glass box above vortices in water, and photographs of the stream-lines at different levels are easily obtained. The most interesting case we have yet observed is that produced when two paddles, rotating in opposite directions in water, set up a surface circulation similar to that in a bipolar spot group. The stream-lines in the smoke, seen when looking vertically downward toward the water, resemble those of a vortex ring of colored liquid rising through water from the orifice of a tube. When observed from a point in the plane of the ring, the stream-lines in the latter case appear to be closed on the upper or advancing side, while the surplus liquid on the following side moves along lines which are straight and axial opposite the center of the ring and become more and more convex on either side of this axis.[6] In the smoke similar stream-lines appear in horizontal planes, producing the effect of a hood

or arch on one side of the line joining the two 'spots,' with the very different structure just described on the opposite or following side.

Such an appearance of asymmetry would seem to be faithfully reproduced on some of our photographs of the hydrogen flocculi surrounding bipolar spot groups. If this resemblance should prove to be other than superficial, it would apparently indicate that in these cases the hydrodynamic influence of the spot vortices outweighs their electromagnetic effect in determining the structure of the flocculi. It may be added that the inward (nearly horizontal) flow of the smoke at high levels, the downward flow at intermediate levels and the outward flow at low levels is in harmony with solar observations.[7] But no final conclusions can be drawn before the experiments have been multiplied, and the necessary magnetic influence introduced, if possible, by means of magnets within the liquid vortices, acting on the moving smoke particles, ionized by an X-ray discharge, and separated electrostatically. As other photographs of the hydrogen flocculi surrounding bipolar spots seem to show a more nearly symmetrical structure, resembling the lines of force about a bar magnet, it may be true that in such cases the electromagnetic influence predominates, since we have no evidence that one spot may be regarded as a source and the other as a sink. The possible effect of electric fields in and about sun-spots must also be considered.[8]

It has been shown in this paper that some of the phenomena of single and multiple sun-spots can be imitated by simple laboratory experiments, which may also assist in explaining certain characteristic structures and motions of the solar atmosphere. The full details of the investigation will be published in the *Astrophysical Journal*.

[1] The vertical tube shown at the bottom of the tank, used for another experiment, has no effect on this result.

[3] See these PROCEEDINGS, 1, 382 (1915).

[3] *Observations of Solar Spots*, p. 222.

[4] *London, Mon. Not. R. Ast. Soc.*, 73, 687 (1913).

[5] Kelvin, *Mathematical and Physical Papers*, vol. 4, p. 67.

[6] See Mach, *Ann. Physik, Leipzig*, 68, Plate III, fig. 13. A better resemblance is afforded by a photograph of a smoke ring by Wood, reproduced in *Nature*, 63, 418, fig. 1.

[7] The stream-lines in different planes, either vertical or horizontal, are most clearly shown by a narrow beam of sunlight, admitted through the glass wall of the tank by means of a slit, which can be set in any desired position. In this way it is easy to imitate phenomena observed with the spectroheliograph in plan on the sun's disk and in elevation at the limb, where Slocum and others have photographed prominences while being drawn into sun-spots. Radial stream-lines in the horizontal plane, like those often indicated by the hydrogen flocculi about single spots, are well shown by the smoke at high levels.

[8] An error occurs in a paper by Hale and Babcock in a recent number of these PROCEEDINGS (March 1915, p. 125). Having in mind the lines of force between the separated charges in the vortex, the electric lines of force in the spot were described as tangential to the solar surface. This statement is more likely to apply only to the edges of the spot. At its center the lines of force are probably nearly radial.

NOTICES OF BIOGRAPHICAL MEMOIRS

The following biographical memoirs have been published by the Academy since the beginning of this calendar year.

JOHN WESLEY POWELL (1834–1902). By W. M. Davis. *Biographical Memoirs of the National Academy*, vol. 8, pp. 1–83.

This Memoir discusses the life-work of its subject under the headings: Early Life; Service in the Civil War; Visit to the Rocky Mountains, 1867–1868; Exploration of the Colorado Canyon; Geological Survey of the Territories; Report on the Colorado Canyon; Antecedent Rivers; Geological Work; Physiographic Work; Baselevel of Erosion; Planation; Physiographic Essays; Lands of the Arid Region; the Geological Survey; Topographical Map; Reports and Folios; Irrigation Survey; Administration; Resignation from the Survey; Residence in Washington; Ethnological Work; Bureau of Ethnology; Indian Languages and Mythology; Savagery, Barbarism, and Civilization; Synthetic Essays; Mannerisms; Views on Evolution; Evolution of Music; Inductive Studies; Indian Linguistic Families; Philosophical Studies; Pentalogic Series; Truth and Error; Personal Estimates.

CHARLES ANTHONY SCHOTT (1826–1901). By Cleveland Abbe. *Biographical Memoirs of the National Academy*, vol. 8, pp. 87–133.

This Memoir is subdivided into sections entitled: Autobiography of the Early Life of Charles Anthony Schott; Bibliography; Hydrography and Office Work; the Computing Division; Longitude and Latitude; Chartography; Miscellaneous Observations; Terrestrial Magnetism; Climatology; Base-lines and Metric Standards; Geodesy; A Catalogue of the Works (Abridged Titles) of Charles Anthony Schott; Unpublished Works.

MIERS FISHER LONGSTRETH (1819–1891). By Rebecca C. Longstreth. *Biographical Memoirs of the National Academy*, vol. 8, pp. 137–140.